$\vec{F} = \dot{p} = \vec{B}\times\vec{V}$

~~$\mathcal{L} = -m\frac{d\tau}{dt} + qA_\mu\frac{dx^\mu}{d\tau}$~~

~~$-m\sqrt{A_\mu x^\mu} + qA_\mu\frac{dx^\mu}{d\tau}$~~

~~$-m\sqrt{1-\dot{x}^2}$~~

halfway through driving the lorenze force law I remembered google

$$\frac{d}{dt}\left(\frac{\dot{x}m}{\sqrt{1-|\dot{x}|^2}}\right) = q(\vec{E}+\dot{x}\times\vec{B})$$

$$\dot{p} = q(\vec{E}+\vec{V}\times\vec{B}) \qquad p = \frac{vm}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\dot{\vec{p}} = q\left(\cancelto{0}{\vec{E}} + \frac{\vec{p}\times\vec{B}}{\sqrt{1+p^2}}\right)$$

$$E = \sqrt{p^2+m^2}$$

$$\dot{p} = \frac{q}{\sqrt{1+\frac{p^2}{c^2}}}(p\times B)$$

$$B = \begin{bmatrix}0\\0\\b\end{bmatrix}, \quad p = \begin{bmatrix}p_x\\p_y\\0\end{bmatrix} \qquad p\times B = \begin{bmatrix}-bp_y\\bp_x\\0\end{bmatrix}$$

$$V^2\left(1+\frac{p^2}{c^2}\right) = p^2$$

$$\vec{V} = \frac{\vec{p}}{\sqrt{1+\frac{p^2}{c^2}}}$$

$$p_x = p_0\cos\left(\frac{qbt}{\sqrt{m^2+\frac{p_0^2}{c^2}}}\right)$$

$$p_y = p_0\sin\left(\frac{qbt}{\sqrt{m^2+\frac{p_0^2}{c^2}}}\right)$$

$$E^2 = p^2c^2+m^2c^4$$

$$\frac{E^2-m^2c^4}{c^2} = p^2$$

$$\frac{E^2}{c^4} - m^2 = \frac{p^2}{c^2}$$

$$p_0 = \frac{1}{c}\sqrt{E^2-m^2c^4}$$

$$\frac{qb}{\sqrt{m^2+\frac{p_0^2}{c^2}}}$$

$$\frac{qb}{\sqrt{\cancel{m^2}+\frac{E^2}{c^4}\cancel{-m^2}}}$$

$$\cancel{p_0}\omega\cos(\cancel{\omega t}) = \frac{qb}{\sqrt{1+\frac{p_0^2}{c^2}}}\cancel{p_0}\cos(\cancel{\omega t})$$

$$p_x = p_0\cos\left(\frac{qcbt}{\sqrt{c^2+p_0^2}}\right) \qquad p_y = p_0\sin\left(\frac{qcbt}{\sqrt{c^2+p_0^2}}\right)$$

$$\vec{V} = \frac{\vec{p}}{\sqrt{1+\frac{p^2}{c^2}}}$$

$$p^2 = v^2\left(m^2+\frac{p^2}{c^2}\right)$$

$$\dot{p}_x = \frac{-qb}{\sqrt{m^2+\frac{p_x^2+p_y^2}{c^2}}}p_y$$

$$\dot{p}_y = \frac{qb}{\sqrt{m^2+\frac{p_x^2+p_y^2}{c^2}}}p_x$$

some important ODE's

everyone feels weird without the (admitedly redundant) "0"

$$p_x = p_0\cos(\omega t) + \qquad p_y = p_0\sin(\omega t)$$

$$-\cancel{p_0}\omega\sin(\cancel{\omega t}) = \frac{+qb}{\sqrt{m^2+\frac{p_0^2}{c^2}}}\left(\cancel{p_0\sin(\omega t)}\right)$$

$$\frac{p}{\sqrt{m^2+\frac{p^2}{c^2}}} = \frac{\frac{p}{m}}{\sqrt{1+\frac{(\frac{p}{m})^2}{c^2}}} = V \qquad \frac{pc}{\sqrt{m^2c^2+p^2}}$$

$$\omega = \frac{qb}{\sqrt{m^2+\frac{p_0^2}{c^2}}}$$

$$\vec{V} = \frac{\vec{p}}{\sqrt{m^2+\frac{p^2}{c^2}}} =$$

$$v_0 = \frac{1}{c}\sqrt{E^2-m^2c^4}\left(\frac{c^2}{E}\right) = \boxed{c\sqrt{1-\frac{m^2c^4}{E^2}} = v_0}$$

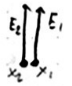
$$V_x = p_0\left(\frac{1}{\sqrt{m^2+\frac{p_0^2}{c^2}}}\right)\cos(\omega t) = v_0\cos(\omega t)$$

$$V_y = p_0\left(\frac{1}{\sqrt{m^2+\frac{p_0^2}{c^2}}}\right)\sin(\omega t) = v_0\sin(\omega t)$$

$$\boxed{\begin{aligned} x(t) &= \frac{v_0}{\omega}\sin(\omega t) + x_0 \\ y(t) &= -\frac{v_0}{\omega}\cos(\omega t) + y_0 \end{aligned}}$$

$$p_x = p_0\cos\left(\frac{qbc^2}{E}t\right)$$

$$p_y = p_0\sin\left(\frac{qbc^2}{E}t\right)$$



at this point I gave up on doing everything by hand and started useing mathematica

$$\boxed{\begin{aligned} V_0 &= c\sqrt{1-\frac{m^2c^4}{E^2}} \\ \omega &= \frac{qbc^2}{E} \end{aligned}}$$

DON'T FORGET THESE DEFINITIONS!

$$x_1(t) = \frac{v_0}{\omega}\sin(\omega t)$$

$$y_1(t) = -\frac{v_0}{\omega}\cos(\omega t) + \frac{v_0}{\omega}$$

$$x_2(t) = \frac{v_0}{\omega}\sin(\omega t)$$

$$y_2(t) = -\frac{v_0}{\omega}\cos(\omega t) + \frac{v_0}{\omega}$$

distance between particles

~~$d(t) = \sqrt{\frac{v_0^2}{\omega^2}\left[(\sin\omega_1 t - \sin\omega_2 t)^2\right] + \frac{v_0^2}{\omega^2}\left[\cos(\omega t) - \cos(\omega_2 t) - \frac{v_1^0}{\omega_1} + \frac{v_2^0}{\omega_2}\right]^2}$~~

$$d(t) = \sqrt{\left(\frac{v_1^0}{\omega_1}\sin(\omega_1 t) - \frac{v_2^0}{\omega_2}\sin(\omega_2 t)\right)^2 + \left(\frac{v_1^0}{\omega_1}[\cos\omega_1 t - 1] - \frac{v_2^0}{\omega_2}[\cos\omega_2 t - 1]\right)^2}$$

~~$d(t) \approx \cancelto{0}{d(0)} + \dot{d}(0)t$~~

cause we're working with times that are tiny compared with all other variables (in the) problem

$$d(t) \approx \dot{d}(0)t + \frac{t^2}{2}\ddot{d}(0)$$

unfortunately we need more terms... t^2 is 0 and $\dot{d}(0)$ isn't usefull not depending on magnetic field!

$\partial_t \vec{\psi} = \hat{H}\vec{\psi}$

$\vec{\psi}(x,t) = \int \hat{K}(x, x_0, t)\psi(x_0, 0)\, dx_0$

$\hat{\dot{x}} = [\hat{x}, \hat{H}]$

$\hat{x}(t)\hat{K}(x, x_0, t) = x_0 \hat{K}(x, x_0, t)$

method for solving linear PDE's

Definition of inner product

$\left\langle \begin{smallmatrix}\text{final}\\ \text{state}\end{smallmatrix} \middle| \begin{smallmatrix}\text{initial}\\ \text{state}\end{smallmatrix} \right\rangle = (\text{amplitude})$

I've had this backward for soooooo long...

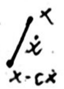
$[x, P] = i\hbar$

$P \Rightarrow -i\hbar\frac{\partial}{\partial x}$

$\int f'g + g'f\, dx = fg$

$\int f'g\, dx = -\int g'f\, dx$

$\langle\psi_1|P|\psi_2\rangle = \int \psi_1^*(x)\left[-i\hbar\frac{\partial}{\partial x}\psi_2(x)\right]dx$



$\int \psi_1^*(x+\varepsilon\dot{x})\psi_2(x)e^{\frac{iS}{\hbar}} Dx$

$\iint \psi_1^*(x+\varepsilon\dot{x})\psi_2(x)\frac{\partial x}{\partial \dot{x}} e^{\frac{i\varepsilon}{\hbar}\mathcal{L}(x-\varepsilon\dot{x}, \dot{x})} dx\, d\dot{x}$

$\iint \psi_1^*(x+\varepsilon\dot{x})\psi_2(x)\frac{\partial x}{\partial \dot{x}} e^{\frac{i\varepsilon}{\hbar}\mathcal{L}(x, \dot{x})} dx\, d\dot{x}$

$\frac{-i\hbar}{\varepsilon}\iint \psi_1^*(x+\varepsilon\dot{x})\psi_2(x)\frac{\partial}{\partial \dot{x}} e^{\frac{i\varepsilon}{\hbar}\mathcal{L}(x, \dot{x})} dx\, d\dot{x}$

~~$+i\hbar\iint \psi_1^{*\prime}(x$~~

$i\hbar\int \psi_1^{*\prime}(x)\psi_2(x)\, dx$

$\int -i\hbar\psi_1^*(x)\psi_2'(x)\, dx$

HAH!
It WORKS!!!

$\langle\psi_2|\psi_1\rangle = \int \psi_2^*(x(t))\psi_1(x(0))e^{\frac{iS[x]}{\hbar}} Dx$

1D fluid flow:

$$\boxed{\begin{aligned}\partial_t v &= -\partial_x p\\ \partial_t p &= -\partial_x v\end{aligned}}$$

$|s\rangle = \begin{bmatrix} v \\ p \end{bmatrix}$ $\mathbb{I} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\partial_t |s\rangle = -\partial_x \mathbb{I} |s\rangle$

$\partial_t |s\rangle = -\mathbb{I}\, \partial_x |s\rangle$

$\partial_t |\hat{s}\rangle = 2i\pi k \mathbb{I} |\hat{s}\rangle$

$|\hat{s}\rangle = e^{2i\pi k \mathbb{I}} |\hat{s}_0\rangle$

$|s(x,t)\rangle = F^{-1}\left[e^{2i\pi k \mathbb{I} t}\right] * |s(x,0)\rangle$

$$e^{2i\pi k \mathbb{I} t} = \frac{1}{2}\begin{bmatrix} e^{2i\pi k t} - e^{-2i\pi k t} & e^{2i\pi k t} + e^{-2i\pi k t} \\ e^{2i\pi k t} + e^{-2i\pi k t} & e^{2i\pi k t} - e^{-2i\pi k t} \end{bmatrix}$$

$$e^{\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} e^{\alpha} + e^{-\alpha} & e^{\alpha} + e^{-\alpha} \\ e^{\alpha} + e^{-\alpha} & e^{\alpha} - e^{-\alpha} \end{bmatrix}$$

$\cancel{\begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix}^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}^n = \begin{cases} \begin{bmatrix} 0 & \alpha^n \\ \alpha^n & 0 \end{bmatrix} & \frac{1}{2}n \in \mathbb{Z} \\ \begin{bmatrix} \alpha^n & 0 \\ 0 & \alpha^n \end{bmatrix} & \frac{1}{2}n \notin \mathbb{Z} \end{cases}$$

$$\frac{1}{2}\begin{bmatrix} \delta(x+t) - \delta(x-t) & \delta(x+t) + \delta(x-t) \\ \delta(x+t) + \delta(x-t) & \delta(x+t) - \delta(x-t) \end{bmatrix} * \begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(v_0(x+t) - v_0(x-t) + \frac{1}{2}(p_0(x+t) + p_0(x-t)) \\ \frac{1}{2}(v_0(x+t) + v_0(x-t) + \frac{1}{2}(p_0(x+t) - p_0(x-t)) \end{bmatrix}$$

$$\boxed{\begin{aligned} v(x,t) &= \tfrac{1}{2}(v(x+t,0) + v(x-t,0) + p(x+t,0) - p(x-t,0)) \\ p(x,t) &= \tfrac{1}{2}(p(x+t,0) + p(x-t,0) + v(x+t,0) - v(x-t,0)) \end{aligned}}$$

complete solution for 1D fluid flow ↗

$\ddot{f} = \alpha f + C$

$\dot{f} - \alpha f = C$

$\dot{f} + 2i\sqrt[2]{\alpha} f - \alpha f$

$\ddot{f} = \alpha f$

$\ddot{g} = \alpha g + C$

$g = f + \frac{C}{2}t^2$

$f + \frac{C}{2}t^{2}$

$\dot{f} + C$

$\alpha \ddot{f}$

$\ddot{f} = -\alpha f + C$

$\ddot{f} + \alpha f = C$

$\ddot{f} + 2\sqrt[2]{\alpha}\dot{f} + \alpha f = C + 2\sqrt[2]{\alpha}\dot{f}$

$e^{\sqrt[2]{\alpha}t}\ddot{f} + 2e^{\sqrt[2]{\alpha}t}\dot{f} + e^{\sqrt[2]{\alpha}t} f = Ce^{\sqrt[2]{\alpha}t} + 2\sqrt[2]{\alpha}\dot{f}e^{\sqrt[2]{\alpha}t}$

$\frac{d^2}{dt^2}(e^{\sqrt{\alpha}t} f) = e^{\sqrt{\alpha}}$

$\ddot{f} +$

1D fluid flow (alternate model):

$\partial_t F = v$

$$\boxed{\partial_t^2 F = -\partial_x^2 F + f}$$

laws α

$\partial_t^2 \hat{F} = -4\pi^2 k^2 \hat{F} + \hat{f}$

$\partial_t v = -\partial_x p$

$\partial_t p = -\partial_x v$

$\partial_t^2 v = -\partial_x^2 v$

$\partial_t^3 F = -\partial_x^2 \partial_t F$

$\partial_t^2 F = -\partial_x^2 F + f(x)$

$P_0 = \det(H F_0)$

$x^2 + bx + c$

$x^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4}$

$(x + \frac{b}{2})^2 + c - \frac{b^2}{4}$

$\ddot{f} + \alpha f = C$

$\ddot{f} = -\alpha f + C$

$\dot{f} = \sqrt[2]{2(-\frac{1}{2}\alpha f^2 + Cf) + D}$

$\dot{f} = \sqrt[2]{-\alpha f^2 + Cf + D}$

$\int \frac{1}{\sqrt[2]{-f^2 + f + D}}\, df = \sqrt[2]{\alpha}t$

$\int \frac{1}{\sqrt[2]{f^2 + Cf + D}}\, df = \sqrt[2]{-\alpha}$

$\int \frac{1}{\sqrt[2]{(f+C)^2 + D - C^2}}\, df = \sqrt[2]{-\alpha}t$

$u = f + C$

$du = df$

$\int \frac{1}{\sqrt[2]{u^2 + D}}\, du = \sqrt[2]{-\alpha}t$

$\int \frac{1}{\sqrt[2]{D - u^2}}\, du = \sqrt[2]{\alpha}t$

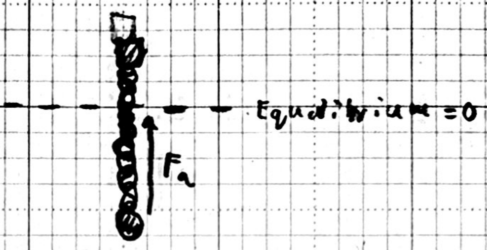
$\int \frac{\frac{1}{\sqrt{D}}}{\sqrt[2]{1 - (\frac{u}{D})^2}}\, du$ $\int \frac{1}{\sqrt[2]{1 - v^2}}\, dv$

$v = \frac{u}{\sqrt[2]{D}}$

$dv = du \cdot \frac{1}{\sqrt[2]{D}}$

$du = \sqrt[2]{D}\, dv$

$\ddot{y} \propto -y$



$\boxed{\ddot{y} \propto + y}$

$\ddot{y} = -ky$

$\dot{y} = \sqrt[2]{2(-\frac{k}{2}y^2 + C)}$

$\dot{y} = \sqrt[2]{-ky^2 + C}$

$\dot{y} = \sqrt[2]{k} \cdot \sqrt[2]{C - y^2}$

$\int \frac{1}{\sqrt[2]{C - y^2}} dy = \int \sqrt[2]{k} dt$

$\int \frac{1}{\sqrt[2]{C}} \cdot \frac{1}{\sqrt[2]{1 - (\frac{y}{\sqrt[2]{C}})^2}} dy = \sqrt[2]{k} t$

$u(x) = \frac{x}{\sqrt[2]{C}}$

$\int \frac{du\ 1}{\sqrt[2]{1-u^2}} \cdot \frac{du}{\cancel{dy}} \cancel{dy} = \sqrt[2]{k} t$

$\int \frac{1}{\sqrt[2]{1-u^2}} du = \sqrt[2]{k} t$

arcsin

$\sin^{-1}(u) + D = \sqrt[2]{k} t$

$\sin^{-1}(\frac{y}{\sqrt[2]{C}}) + D = \sqrt[2]{k} t$

$\frac{y}{\sqrt[2]{C}} = \sin(\sqrt[2]{k} t)$

$$\boxed{y(t) = \sqrt[2]{C} \sin(\sqrt[2]{k}\, t + D)}$$
general solution

$y(t) = \sqrt[2]{\frac{v_0^2}{k}}$ [illegible] $\sin(\sqrt[2]{k} t + D)$

$y(t) = \sqrt[2]{C} \sin(\sqrt[2]{k} t + D)$

$y'(t) = \sqrt[2]{kC} \cos(\sqrt[2]{k} t + D)$

$y_0 = \sqrt[2]{C} \sin(D)$

$v_0 = \sqrt[2]{kC} \cos(D)$

$y_0^2 = C \sin^2(D)$

$\sin^2(D) = \frac{1 - \cos(2D)}{2}$

$2y_0^2 = C - C\cos(2D)$

$2y_0^2 - C = -C\cos(2D)$

$C - 2y_0^2 = C\cos(2D)$

$1 - \frac{2}{C} y_0^2 = \cos(2D)$

$D = \frac{\cos^{-1}(1 - \frac{2}{C} y_0^2)}{2}$

$2v_0^2 = kC + kC\cos(2D)$

$2v_0^2 = kC + kC(1 - \frac{2}{C} y_0^2)$

$2v_0^2 = kC + kC - 2ky_0^2$

$\cancel{2}v_0^2 = \cancel{2}kC - \cancel{2}ky_0^2$

$kC = v_0^2 + ky_0^2$

$$\boxed{C = \frac{v_0^2}{k} + y_0^2}$$

$1 - \frac{2}{C} y_0^2 = 1 - \frac{2}{\frac{v_0^2}{k} + y_0^2}$

~~$y(t) = \sqrt[2]{\frac{v_0^2}{k} + y_0^2} \sin(\sqrt[2]{k} t + \frac{\cos^{-1}(1 - \frac{2}{C} y_0^2)}{2})$~~

~~potential (and very painful) taylored solution~~

$y(t) = \sqrt[2]{\frac{v_0^2}{k} + y_0^2} \sin(\sqrt[2]{k} t + \frac{1}{2}\cos^{-1}(1 - \frac{2}{\frac{v_0^2}{k} + y_0^2} y_0^2))$

even more painful solution

$g : \{ f : \mathbb{R}^2 \to \mathbb{R}^1 \} \to \mathbb{R}^1$

∇ d d d d ∇

$g[f]$

∇_f

$\nabla_f g[f]$

$$g[f] = \int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy$$

$$\nabla_f g[f] := \underbrace{\frac{d}{dq}\int_{-1}^{1}\int_{-1}^{1} f(x,y) + q\,\delta(x-\alpha, y-\beta)\,dx\,dy \Big|_{q=0}}$$

$$\boxed{\nabla_f g[f](\vec{\alpha}) := \partial_q\, g[f(\vec{x}) + q\,\delta(\vec{x}-\vec{\alpha})]\Big|_{q=0}}$$

$$\frac{d}{dt}[f(t) + \delta(t-\alpha)] = f'(t) + \delta'(t-\alpha)$$

$$\cancelto{0}{\frac{d}{dq}\int\int_{-1}^{1} f(x,y)\,dx\,dy} + \cancelto{1}{\frac{d}{dq}\,q}\; \cancelto{1}{\int_{-1}^{1}\int_{-1}^{1}\delta(x-\alpha, y-\beta)}$$

$$\boxed{\nabla_f g[f] = 1}$$

and this works too!

$f : \mathbb{R}^1 \to \mathbb{R}^1$

$$\nabla_f g[f, f'] := \partial_q g[f(t) + \delta(t-\alpha), f'(t) + \delta'(t-\alpha)]\Big|_{q=0}$$

stationary point: $\nabla_f g[f] \equiv 0$

$t \in [a,b]$

$$\int_a^b f(\tau)\,\delta'(\tau - x)\,d\tau = -f'(x)$$

$$\lim_{h\to 0}\int_a^b \frac{f(\tau)\,\delta(\tau - x + h) - f(\tau)\,\delta(\tau - x)}{h}$$

$$\lim_{h\to 0}\frac{f(x-h) - f(x)}{h} = -f'(x)$$

$$A(x) = \int_a^b L(x, \dot{x})\,dt$$

$$\nabla_x A(x) \equiv 0$$

$$\cancel{\partial_q}\int_a^b L(x(\tau) + q\,\delta(\tau - t), \dot{x}(\tau) + q\,\dot{\delta}(\tau - t))\,d\tau$$

$$\partial_q \int_a^b L(x(\tau) + q\,\delta(\tau - t), \dot{x}(t) + q\,\dot{\delta}(\tau - t))\,d\tau\,\Big|_{q=0}$$

$$\int_a^b \frac{\partial L}{\partial x}\cdot\delta(\tau - t) + \frac{\partial L}{\partial \dot{x}}\cdot\dot{\delta}(\tau - t)\,d\tau = 0$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}}$$

yippy! It works!