<u>MARK (MAX) ORTON – STEM PORTFOLIO</u>

This STEM portfolio contains 5 write-ups showcasing some of my physics and math work over the last three years. I've been studying quantum mechanics, quantum field theory, partial differential equations, and some general relativity. This work has been almost completely independent. I've learned most of it through textbooks, google searches and a lot of freewheeling experimentation.

Much of what I've learned has come from "The Theoretical Minimum," a series of books and recorded lectures by Leonard Susskind. As I read and watched TTM, I challenged myself to never go on to the next section without mastering the math behind it. Several of the write-ups here are my attempts to expand on topics Susskind mentioned in passing, but never fully explained.

I absolutely love deriving these equations. I've always been amazed that with just a notebook and a pencil, we can work out the properties of phenomena in the world around us that we've never even seen. It feels like magic.

Other than a little help from my high school physics teacher to get me up to speed on LaTeX, I've created these write-ups entirely on my own. I don't have any experience writing scientific papers but have done my best to convey my process and results.

I've been TA'ing math and physics and founded the math club at my school, but I haven't found anybody with a strong enough interest in theoretical physics to bounce ideas off. I'm looking forward to attending a school where I have the possibility of studying advanced physics and doing some research as an undergraduate, joining a community of people whose passion for physics and math is as strong as mine.

Modeling Relativistic Strings as Level Curves

Max Orton

Fall 2024

My work with Relativistic Strings is my most recent project. In The Theoretical Minimum recorded lectures on String Theory, Susskind used parameterized strings, and worked in the lightcone frame. I wanted to create a manifestly relativistic model that could handle multiple strings at the same time. This write-up is the result.

Our goal in the following is to find a way to describe 1+1 dimensional structures consistent with special relativity. We will begin in 2+1 dimensions, then generalize to higher dimensions and discuss possible ways to quantize our system.

Our starting point will be the fact that for a collection of relativistic strings, the action is proportional to the total surface area of the collection. Though later we will find a more explicit form for the action, for now we can write:

$$S = kA$$

Where S is the action, A is the total surface area, and k is a constant to maintain dimensional consistency. It's worth noting that k will need units of action divided by area which means that k is a force or, in the same vein, a tension. This works because, though area normally has units of length squared, a string moving slower than light will instead have its area measured in units of length time. When I first noticed this, I was very excited, but honestly, I think this result might just be a coincidence. We will encounter a better physical interpretation for k later on.

There is only so much we can do with the action written in so abstract a form, so our next step will be to write it explicitly in terms of the model we are using to describe the strings. We will be modeling the strings as the level curves of, in general, D-1 functions in D+1 dimensional space. The advantage to this approach is that multiple strings and even interactions between strings are baked into the model's structure and do not have to be added later.

2+1 dimensional space is the simplest, non-trivial example, so we will begin there. To do this, we will define a function $\phi(X)$ such that at each point X in space time where a string exists:

$$\phi(X) = 0$$

Our next goal will then be to find the surface area of the level curves of this function. We only want points where the function is zero to contribute, so we can try including a delta function of the field. We multiply this by an as yet undetermined function of the derivatives of the field. It is only a function of the derivatives of ϕ, because everywhere the integrand is non-zero, the field is zero.

$$A[\phi] = \int_{t_i}^{t_f} \int_{\Omega} \Lambda(\partial\phi)\delta(\phi)\, dA\, dt$$

We can now try to find the form of Λ. If the field didn't change over time ($\dot{\phi} = 0$), then our integral would simplify:

$$A[\phi] = (t_f - t_i) \int_{\Omega} \Lambda(\nabla\phi)\delta(\phi) dA$$

The integral is evaluated now only on a single time slice and can be seen as the total length of the collection of strings on that slice. That, multiplied by our change in time ($t_f - t_i = \Delta t$), gives us the total relativistic surface area of the strings.

Next, we will break up space into many small regions $\{\Omega_n\}$ and assign each one a point $\vec{x_n} \in \Omega_n$. Each of the Ω_n is picked to be small enough that ϕ is approximately linear within the region.

$$\phi(\vec{x}) \approx \phi(\vec{x_n}) + (\vec{x} - \vec{x_n}) \cdot \vec{\nabla}\phi(\vec{x_n}) \quad \forall x \in \Omega_n$$

We can rewrite our integral as a sum over the individual regions. We will also define shifted integration variables $\vec{y} = \vec{x} - \vec{x_n}$.

$$A[\phi] = \Delta t \sum_n \int_{\Omega_n} \Lambda(\nabla\phi(x_n + y))\delta(\phi(\vec{x_n}) + \vec{y} \cdot \vec{\nabla}\phi(\vec{x_n}))dy_1 dy_2$$

We can now evaluate each of these integrals separately. We will invent a new pair of coordinates that are rotated such that one, $y_\parallel$, is parallel to $\vec{\nabla}\phi$ and the other,$y_\perp$, is perpendicular to it. Since all we did was rotate the coordinates in space, $dy_1 dy_2 = dy_\parallel dy_\perp$. The dot product of y with $\vec{\nabla}\phi$ is simply $y_\parallel \|\vec{\nabla}\phi\|$. Plugging this in, we get:

$$A[\phi] = \Delta t \sum_n \int_{\Omega_n} \Lambda(\nabla\phi(x_n + y))\delta(\phi(\vec{x_n}) + y_\parallel \|\vec{\nabla}\phi\|) dy_\parallel dy_\perp$$

Carrying out the integral over $y_\parallel$ yields:

$$A[\phi] = \Delta t \sum_n \int \frac{\Lambda(\nabla\phi(x_n + y))}{\|\vec{\nabla}\phi\|} dy_\perp \,|_{y_\parallel = 0}$$

The integral is performed along the small segment of the curve $\phi = 0$ contained inside Ω_n. Adding all our regions back together this gives us a line integral over the curve $\phi = 0$.

$$A[\phi] = \Delta t \oint_{\phi(\vec{x})=0} \frac{\Lambda(\nabla\phi)}{\|\vec{\nabla}\phi\|} ds$$

If we want the integral to represent the length of the curve, the integrand should clearly be 1. This finally allows us to pin down a form for Λ!

$$\Lambda(\nabla\phi) = \|\vec{\nabla}\phi\|$$

This gives us an explicit action for the case where $\dot{\phi} = 0$:

$$S[\phi] = \int k\Delta t \|\vec{\nabla}\phi\| \delta(\phi) d^2x$$

Finding the general form turns out to be surprisingly easy. There is really only one way to write a relativistically invariant function of $\partial_\mu\phi$ that reduces to $\|\vec{\nabla}\phi\|$ when $\dot{\phi} = 0$. Here, we are using the mostly negative form of $\eta_{\mu\nu}$.

$$\sqrt{\|\vec{\nabla}\phi\|^2 - \dot{\phi}} \quad \text{or equivalently} \quad \sqrt{-\partial_\mu\phi\partial^\mu\phi}$$

Plugging this in, we've found the general form for our action!

$$S[\phi] = \int k\sqrt{-\partial_\mu\phi\partial^\mu\phi}\, \delta(\phi) dX$$

Before we dive into exploring the properties of this action, we'll do two quick example calculations to see if they give reasonable results. The first example is that of a string stretched along the x_2-axes from $-\infty$ to $+\infty$ and moving along the x_1 axes. Since our x_1 coordinate is changing as a function of time, we can write:

$$X(t) = x_1 \quad \text{or} \quad 0 = X(t) - x_1$$

This has the form $\phi(x_1, t) = 0$ which means that $X(t) - x_1$ can serve as ϕ in our model. All we need to do to find $X(t)$ is plug it into our formula for the action.

$$-\partial_\mu \phi \partial^\mu \phi = 1 - \dot{X}^2$$

$$S[X] = \int k\sqrt{1 - \dot{X}^2}\, \delta(X(t) - x_1) dx_1 dx_2 dt$$

Integrating over x_1 and moving around our integral over x_2 we get:

$$S[X] = \int \left(\int k \, dx_2 \right) \sqrt{1 - \dot{X}^2} \, dt$$

This is just the action for a relativistic particle of total mass $-\int k \, dx_2$! This gives us our better physical meaning for the constant k. It is the negative of the mass density of the strings $k = -\mu$. From here on, I will be writing the action in terms of μ instead of k as it has more physical meaning.

We will do one more slightly less trivial example: that of a circular string. We will do something very similar to what we did before when defining $X(t)$, but now we will define $R(t) = \sqrt{x_1^2 + x_2^2} = r$ which implies $\phi(r, t) = R(t) - r$. We can then plug this in to find the action. I have skipped the intermediate steps and jumped right to the final result.

$$S[R] = \int -\mu R\sqrt{1 - \dot{R}^2} \, dt$$

We can then use the Euler Lagrange Equations to find R.

$$\frac{d}{dt}\left(\frac{R\dot{R}}{\sqrt{1 - \dot{R}^2}} \right) + \sqrt{1 - \dot{R}^2} = 0$$

When I first tried this problem, I used conservation of energy to make it first order, then went about solving it mechanically. It was anything but easy. The simpler way is to notice that the form of $\sqrt{1-\dot{R}^2}$ seems to lend itself to a solution in terms of sine or cosine. I have since come up with a third way of solving this, by far my favorite, which includes a Wick rotation and the fact that hanging strings make hyperbolic cosine graphs. This is included in my video. For now we'll go with the second method, plug in $R = \alpha \sin(\beta t + \gamma)$ and see what we get (I'm defining $\theta = \beta t + \gamma$ for the sake of fitting the equation on the page).

$$\frac{d}{dt}\left(\frac{\alpha^2\beta\sin(\theta)\cos(\theta)}{\sqrt{1-\alpha^2\beta^2\cos(\theta)^2}}\right) + \sqrt{1-\alpha^2\beta^2\cos(\theta)^2} = 0$$

This is satisfied if $\alpha\beta = 1$. Thus we have:

$$R(t) = \alpha\sin\left(\frac{t}{\alpha} + \gamma\right)$$

This is a two parameter family of solutions to a second order differential equation, so we can be reasonably confident it represents the entire solution set. There are a few interesting things to note about this solution. First of all, its physical characteristics are determined by only a single number α: the maximum radius. The other constant, γ, is a purely mathematical construct related to when we start counting time.

Second of all, it is pretty clear that the solution only makes sense for $t \in [-\alpha\gamma, \alpha(2\pi + \gamma)]$. Outside of that interval, our solution gives a negative radius. At the point when the radius equals zero, the string's inward velocity $\dot{R}$ approaches the speed of light. This is a kind of singularity. The total energy of the string must stay constant, while the radius (and thus the length) of the string approaches zero. This means that the energy density of the string blows up to infinity as the string shrinks.

Over the last month or so I've been trying to show that this singularity happens for all collections of strings of finite size in 2+1 dimensional space, or that there are some configurations that avoid it. For now, I haven't been able to show either. My attempt has centered around checking whether the area encompassed by the collection of strings must shrink to zero. The change in area does have a very simple form: the integral of $\dot{\phi}\delta(\phi)$ over all space. I haven't yet been able to show what this implies.

For now, we can use the Euler Lagrange equations to find a general equation of motion for our system.

$$\frac{\partial}{\partial X^\mu}\frac{\partial \mathcal{L}}{\partial\left[\frac{\partial\phi}{\partial X^\mu}\right]} = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{\partial}{\partial X^\mu}\left[\frac{-\partial_\mu\phi\,\delta(\phi)}{\sqrt{-\partial_\mu\phi\partial^\mu\phi}}\right] = \sqrt{-\partial_\mu\phi\partial^\mu\phi}\,\delta'(\phi)$$

Amazingly, when we expand it, all the terms multiplying $\delta'(\phi)$ cancel, leaving:

$$[\partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi - \partial_\sigma\phi\partial_\sigma\phi\partial^\tau\partial_\tau\phi]\delta(\phi) = 0$$

There are multiple ways to satisfy this equation. Perhaps the easiest is to constrain ϕ more, restricting ourselves to ϕ for which the part multiplying the delta function is zero at all points, not just when the delta function itself is non-zero. This is still a second order, non-linear partial differential equation. It is anything but easy to solve.

So far, we have only studied strings in 2+1 dimensions but I'll conclude by generalizing our action to D+1 dimentional space. The action will now be a functional of D-1 functions $\{\phi_n\}$. The string exists at all points X where:

$$\phi_n(X) = 0 \quad \forall n$$

Just like before, we only want points where the string exists to contribute to the action, so we will write it as a delta function multiplied by a function of the derivatives of all the ϕ_n.

$$S = \int_{t_i}^{t_f}\int_\Omega \Lambda(\{\partial\phi_n\})\prod_n \delta(\phi_n)\,dx_i\,dt$$

Like last time, we first consider the case where the string is stationary ($\dot{\phi} \equiv 0 \;\; \forall n$).

$$S = \Delta t \int_\Omega \Lambda(\{\nabla\phi_n\})\prod_n \delta(\phi_n)\,dx_i$$

Then, we break up omega into many sub-regions Ω_α (small enough that all the ϕ_α are approximately linear over the regions) and pick corresponding points $x_\alpha \in \Omega_\alpha$, then split up our integral.

$$S = \Delta t \sum_\alpha \int_{\Omega_\alpha} \Lambda(\{\nabla\phi_n(x)\})\prod_n \delta(\phi_n(x))\,dx_i$$

Like last time, we define new integration variables in each of our regions $(\{y_{\|n}\}, y_\perp)$. These are shifted so that the new origin corresponds to x_α, and all but one of the unit vectors (unit both in the new and old system) point along the D-1 gradients of the ϕ_n, while the last one is perpendicular to all of them. We can write our new volume element in terms of the derivatives of ϕ where ϵ is the completely antisymmetric tensor.

$$dx = \frac{\partial\{x_i\}}{\partial\{y_i\}} dy_\| \, dy_\perp = \frac{\prod_n \|\nabla\phi_n\|}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}} dy_\| \, dy_\perp$$

Next we will plug this into our integral and simultaneously make the assumption that all the ϕ_n are approximately linear in the regions Ω_α. This gives us:

$$S = \Delta t \sum_\alpha \int_{\Omega_\alpha} \Lambda(\{\nabla\phi_n(x)\}) \prod_n \delta(\|\nabla\phi_n\| y_{\|n}) \, \frac{\prod_n \|\nabla\phi_n\|}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}} \, dy_\| \, dy_\perp$$

Integrating over all of the $y_{\|n}$ cancels the product in the numerator of our volume element, giving us:

$$S = \Delta t \sum_\alpha \int_{\Omega_\alpha, \phi=0} \frac{\Lambda(\{\nabla\phi_n(x)\})}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}} \, dy_\perp$$

Each of these is an integral over the small and approximately straight segment of string contained in each of the Ω_α. I haven't explicitly written it out, but if there was no point in a given Ω_α where $\phi_n = 0 \ \forall n$, then the integrand for that Ω_n would instead be zero. Like last time, adding all of the Ω_α back together yields a line integral over the strings.

$$S = \Delta t \oint_{\phi(\vec{x})=0} \frac{\Lambda(\{\nabla\phi_n\})}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}} \, ds$$

If we want this to be proportional to the total length of all the strings, the integrand needs to be constant. We can write it in terms of μ, the mass density of the string. This finally lets us pin down a form for Λ!

$$\Lambda(\{\nabla\phi_n\}) = -\mu \sum_{\{i_n\}} \epsilon_{\{i_n\}} \prod_n \frac{\partial\phi_n}{\partial x_{i_n}}$$

We can plug this into our action in terms of Λ to find the action of a stationary string.

$$S = \Delta t \int_{\Omega} -\mu \sum_{\{i_n\}} \epsilon_{\{i_n\}} \prod_{n} \frac{\partial \phi_n}{\partial x_{i_n}} \delta(\phi_n) d^3x$$

It's much less obvious than last time, but there is still basically only one relativistically invariant function of the derivatives of the ϕ_n that reduces to the above integrand when $\dot{\phi}_n = 0$ (The first picture on my website is me showing how this reduces to the above action for a stationary string). Substituting this into our integral, we've found the total action for a collection of relativistic strings in D+1 dimensional space!

$$S = \int_{t_i}^{t_f} \int_{\Omega} -\mu \sqrt{\epsilon^{\mu\{\beta_n\}} \epsilon_{\mu\{\alpha_n\}} \prod_{n}^{D-1} \frac{\partial \phi_n}{\partial x_{\alpha_n}} \frac{\partial \phi_n}{\partial x^{\beta_n}}} \prod_{n}^{D-1} \delta(\phi_n) d^3x$$

If we solve for our equations of motion using the Euler Lagrange equations, we again find that all the terms containing $\delta'(\phi_n)$ cancel, and we get a single equation multiplied by a delta function of the field.

$$\forall j \quad \left[\prod_{n} \delta(\phi_n)\right] \frac{\partial}{\partial x_{\alpha_j}} \left[\frac{\epsilon^{\mu\{\beta_n\}} \epsilon_{\mu\{\alpha_n\}} \left(\prod_{n \neq j}^{D-1} \frac{\partial \phi_n}{\partial x_{\alpha_n}} \frac{\partial \phi_n}{\partial x^{\beta_n}} \right) \frac{\partial \phi_j}{\partial x^{\beta_j}}}{\sqrt{\epsilon^{\mu\{\beta_n\}} \epsilon_{\mu\{\alpha_n\}} \prod_{n}^{D-1} \frac{\partial \phi_n}{\partial x_{\alpha_n}} \frac{\partial \phi_n}{\partial x^{\beta_n}}}} \right] = 0$$

If we mandate that everywhere, not just $\phi_n = 0$, the field follows this equation, we can write the evolution of ϕ as a field equation with no delta functions. It's worth remarking how amazing it is that for all dimensions, terms containing derivatives of δ cancel. This was far from what I expected. So far this is the only action I've found that includes a delta function where this happens. Proving that this is the only action where the higher order derivatives of delta functions cancel is another project I've been working on.

Though everything written here was done classically, in the future I hope to quantize this model. I've been working on this from two angles. The first is to plug the action (delta function and all) into the path integral. The second is to treat the field equation I derived as a quantum field, then look for level curves using a delta function of the field operator. I've also been looking into representing higher dimensional shapes and open strings with similar models.

Perturbative Expansion for Interacting Quantum Field Theories

Max Orton

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The work here is, in many ways, an extension of my quantization of free fields. Though my original method worked well with linear fields, it couldn't handle interactions, so I spent a year developing this perturbative approach. I find it amazing that this, combined with my quantization of linear fields, describes nearly all the matter within and around us.

Given the free field evolution for a set of bosonic and fermionic fields and a form for the interaction terms in the action, our goal to predict the evolution of the fields.

The amplitude that an initial state $|\psi_I\rangle$ will transform into a final state $|\psi_F\rangle$ after an elapsed time T is:

$$\langle \psi_F, T | \psi_I \rangle = \langle \psi_F | \, e^{-iTE} \, | \psi_I \rangle$$

In order to explore theories where only one part of the Hamiltonian is known we make the following definition:

$$E = E_F(0) - L_I(0)$$

Here, E_F governs the free evolution of the system, while L_I is the interacting terms in the Lagrangian. The total Hamiltonian E is time independent, but in general, E_F and L_I will not be conserved, and so we have arbitrarily chosen to evaluate them at time 0.

We can next use one definition of e^x to expand our time evolution:

$$\langle\psi_F| e^{-iTE} |\psi_I\rangle = \lim_{\epsilon\to 0} \langle\psi_F| [1 - i\epsilon E_F(0) + i\epsilon L_I(0)]^{(\frac{T}{\epsilon})} |\psi_I\rangle$$

We can now expand this to increasing order in the interacting Lagrangian. Our first term, $\mathcal{O}(L_I^0)$, will of course simply be:

$$\lim_{\epsilon\to 0} \langle\psi_F| [1 - i\epsilon E_F(0)]^{(\frac{T}{\epsilon})} |\psi_I\rangle = \langle\psi_F| e^{-iTE_F(0)} |\psi_I\rangle$$

This just describes evolution in the free theory, the details of which we already know. Next, we will consider the terms in the expansion that contain one power of L_I. These will be characterized by a single number detailing which factor in the product it was pulled from. We can call this number n and associate it with a time $t \in [0, T]$ where $t = \epsilon n$. Using t gives us one distinct advantage: while n depends on our choice of discretization, t does not, so it will still make sense after ϵ tends to zero. This second term in our expansion will have the following form:

$$\lim_{\epsilon\to 0} \sum_n \langle\psi_F| (1 - i\epsilon E_F(0))^{\frac{T}{\epsilon}-n} i L_I(0)(1 - i\epsilon E_F(0))^{n-1} |\psi_I\rangle$$

Taking the limit as ϵ tends towards 0, and remembering our definition of t, we get:

$$i \int_0^T \langle\psi_F| e^{-iTE_F(0)} \left[e^{itE_F(0)} L_I(0) e^{-itE_F(0)}\right] |\psi_I\rangle \, dt$$

The important thing to note is that the section to the left of the brackets is ψ_F shifted to time T in the free theory. Likewise, the section inside the brackets is the Heisenberg formula for operator evolution evaluated on L_I and thus is $L_I(t)$. Bringing this all together we get:

$$\langle\psi_F, T| i \int_0^T L_I(t) dt \, |\psi_I, 0\rangle \;\text{ or }\; \langle\psi_F, T| iS_I |\psi_I, 0\rangle$$

Here, S_I is the interacting part of the action. It's worth noting that S_I is the interacting part of the action for the interval of time we are studying, so is integrated only from the initial to final time.

The derivation of the remaining terms runs along the exact same lines, so I'll only include the results below. Note that $\mathcal{T}$ denotes the time ordered product and everything is evaluated in the free theory:

$$\mathcal{O}(L_I^2) : \int_0^T \int_0^{t_2} \langle\psi_F, T| \left[iL_I(t_2)\right] \left[iL_I(t1)\right] |\psi_I, 0\rangle \, dt_1 dt_2 = \langle\psi_F, T| \mathcal{T} \left\{ \frac{(iS_I)^2}{2} \right\} |\psi_I, 0\rangle$$

$$\mathcal{O}(L_I^3) : \langle\psi_F, T| \mathcal{T} \left\{ \frac{(iS_I)^3}{3!} \right\} |\psi_I, 0\rangle$$

$$\mathcal{O}(L_I^4) : \langle\psi_F, T| \mathcal{T} \left\{ \frac{(iS_I)^4}{4!} \right\} |\psi_I, 0\rangle$$

The general form is clearly:

$$\mathcal{O}(L_I^n) : \langle\psi_F, T| \mathcal{T} \left\{ \frac{(iS_I)^n}{n!} \right\} |\psi_I, 0\rangle$$

Adding all of them up and using the power series for e^x we find a form for the amplitude. Since everything is evaluated in the *free* theory, we know how to calculate it!

$$\langle\psi_F, T| \mathcal{T} \left\{ e^{iS_I} \right\} |\psi_I, 0\rangle$$

I've been searching for a way to quantize interacting fields since hearing about the Feynman calculus in one of Susskind's recorded lectures two years ago. Working this out was one of the most exciting moments of the last few years for me. I also figured out the Feynman propagator as a time ordered product of fields at around the same time, but didn't have time to write it into this portfolio. I haven't yet been able to do many calculations with this method. This is because I'm fairly certain the vacuum state is different in interacting and free theories, which has kept me from correctly specifying initial and final conditions. I'm working on a way to fix that. In the meantime, it's incredible to be able to plug a temporary source field (eg; a lightbulb) into the vacuum and calculate on average how many particles it produces and in what states!

Quantizing a Klein Gordon Field

Max Orton

Fall 2023 through Spring 2024

I first learned Quantum Mechanics through Leonard Susskind's Theoretical Minimum books. It was fascinating, but I wasn't entirely satisfied with it. I already knew the basics of special relativity when I read the book, and the quantum mechanics that Susskind described was inherently non-relativistic. I had an intense curiosity about how to make quantum mechanics relativistic. This began both my work in the Path Integral (see my other write-up, "Deriving the Canonical Quantization Approach from the Path Integral") and Quantum Field Theory. Early on, I tried to find a formula for how the fields would evolve in terms of the action. Susskind had already said that free fields followed wave equations. Unfortunately, I found many many permutations of the action, state vectors, and the vacuum vector that all reduced to the equation he had given when applied to a free field. In the end, I just brute-forced the problem, trying each equation until one started giving reasonable results. It's worth noting that everything here is work I did, and wrote up, in my Junior year. I have since found more efficient methods for some of this. Still, I thought I'd include it as a kind of "time capsule" of what I was obsessed with in eleventh grade.

We begin by defining a collection of operators (note here $c = \hbar = 1$):

$$\phi_n(t), \pi_n(t) \quad | \quad \forall t \quad [\phi_n(t), \pi_m(t)] = i\delta_{nm}$$

We will also assume the Lagrangian ($\mathcal{L}$) and Hamiltonian ($\mathcal{H}$) have the following form (here $V(\{\phi\})$ is an arbitrary potential that depends on all the ϕs):

$$\mathcal{L} = \sum_n \frac{1}{2}\dot{\phi}_n^2 - V(\{\phi\}), \quad \mathcal{H} = \sum_n \frac{1}{2}\dot{\phi}_n^2 + V(\{\phi\})$$

For this particular choice of Lagrangian and Hamiltonian we are going to prove that the following familiar equation is equivalent to Heisenberg's equation for the time dependence of an operator.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} - \frac{\partial \mathcal{L}}{\partial \phi_n} = 0 \quad \leftrightarrow \quad \dot{L} = i[H, L]$$

Though this at first glance might not seem general enough to be useful this is the form of both the Klein Gordon and Proca Lagrangians. For the sake of this proof we are going to assume that time evolution unfolds according to some unitary operator which is a function of all the ϕ_ns and π_ns:

$$\phi_n(t) = e^{itU(\{\phi\},\{\pi\})}\phi_n(0)e^{-itU(\{\phi\},\{\pi\})}, \quad \pi_n(t) = e^{itU(\{\phi\},\{\pi\})}\pi_n(0)e^{-itU(\{\phi\},\{\pi\})}$$

We are trying to show that, given the equation similar to stationary action above:

$$U(\{\phi\},\{\pi\}) = H(\{\phi\},\{\pi\})$$

First we write out the least action equation more explicitly in terms of the Lagrangian. We will divide it into two different equations, one to define the derivative of the generalized coordinate and the other to define the derivative of the momenta:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} - \frac{\partial \mathcal{L}}{\partial \phi_n} = 0 \quad \rightarrow \quad \dot{\pi}_n = -\frac{\partial V}{\partial \phi_n}, \quad \dot{\phi}_n = \pi_n$$

First, we need to write the first derivative of ϕ and π in terms of U (here ψ refers to an arbitrary operator, ϕ or π as both have the same time dependence and ϵ is a infinitesimal change in time small enough that it's higher powers are negligible):

$$\psi(t) = e^{itU}\psi_n(0)e^{-itU}$$

$$\psi(t+\epsilon) = \psi(t) + \epsilon\dot{\psi}(t) = e^{i\epsilon U}(e^{itU}\psi_n(0)e^{-itU})e^{-i\epsilon U}$$

We see that the interior of the parenthesis is just the definition of $\psi(t)$.

$$\psi(t) + \epsilon\dot{\psi}(t) = e^{i\epsilon U}\psi(t)e^{-i\epsilon U}$$

Expanding out the exponentials to first order in ϵ we get

$$\psi(t) + \epsilon\dot{\psi}(t) = (1 + i\epsilon U)\psi(t)(1 - i\epsilon U)$$

$$\psi(t) + \epsilon\dot{\psi}(t) = \psi(t) + i\epsilon U\psi(t) - i\epsilon\psi(t)U$$

Identifying $\dot{\psi}$ with the component of the right hand side of the expression proportional to ϵ we see that

$$\dot{\psi}(t) = i[U, \psi(t)]$$

This in turn means

$$\dot{\pi}_n(t) = i[U, \pi_n(t)], \quad \dot{\phi}_n(t) = i[U, \phi_n(t)]$$

To show that the Euler Lagrange equations above are equivalent to the Heisenberg time dependent operator equations we need to show that U is equal to the Hamiltonian. To do this we write out the component of the Euler Lagrange equations defining the change in π in terms of U:

$$\dot{\pi}_n = -\frac{\partial V}{\partial \phi_n} \quad \rightarrow \quad [U, \pi_n] = i\frac{\partial V}{\partial \phi_n}$$

Next we use the following equation which follows from the definition of the canonical commutator:

$$[F(\{\phi, \pi\}), \pi_n] = i\frac{\partial F}{\partial \phi_n}$$

Plugging this in and noting that U is a function of all the ϕ's and π's we get:

$$\frac{\partial U}{\partial \phi_n} = \frac{\partial V}{\partial \phi_n}$$

From this it follows that U has the following form where $K(\{\pi\})$ is an arbitrary function which depends only on the π's:

$$U(\{\pi, \phi\}) = K(\{\pi\}) + V(\{\phi\})$$

This is exactly the form we expect to see if $U = \mathcal{H}$: the potential energy plus something that depends only on the π's.

In order to pin down the form of K we need to use the second part of the Euler Lagrange equations, the part that defines the change in ϕ:

$$\dot{\phi}_n = \pi_n \quad \rightarrow \quad -i[\phi_n, U] = \pi_n$$

We need to use another equation that follows directly from the canonical, commutators:

$$[\phi_n, F(\{\phi, \pi\})] = i\frac{\partial F}{\partial \pi_n}$$

Plugging this in we get:

$$\frac{\partial U}{\partial \pi_n} = \frac{\partial K}{\partial \pi_n} = \pi_n$$

Up to a redundant additive constant this means K has the form:

$$K(\{\pi\}) = \sum_n \frac{1}{2}\pi_n^2$$

Plugging this into U we get:

$$U = K + V = \sum_n \frac{1}{2}\pi_n^2 + V(\{\phi\}) = \mathcal{H}$$

Thus:

$$U = \mathcal{H}$$

Now that we have shown the Euler Lagrange equations are equivalent to Heisenberg's equations for the time evolution of an operator, we can apply them to a simple quantum field theory. In this case we will use a Klein Gordon field with mass m. As before we work in natural units: $c = \hbar = 1$. The action is defined as follows (here the integral is over all four dimensions of space-time):

$$S = \int \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2) d^4X$$

Before we can leverage the Euler Lagrange equations on this field we need to do two things. The first is to derive the more useful field theoretic version of the Euler Lagrange equations from the originals above and the second is to justify our choice for the fields canonical commutators.

The derivation of the field theory Euler Lagrange equations from the originals is rather straight forward. First assume that the Lagrangian is the spatial integral over a Lagrangian density (note that from here on, the Lagrangian will be written L while the Lagrangian density will be $\mathcal{L}$):

$$L(t) = \int \mathcal{L}(\{\phi(\vec{x})\}, t) d^3x$$

We then write out the Euler Lagrange Equations using L:

$$\frac{\partial}{\partial t}\frac{\delta L}{\delta \dot{\phi}} - \frac{\delta L}{\delta \phi} = 0$$

Assuming that $\mathcal{L}(\{\phi, \partial\phi\})$ depends only on first derivatives of ϕ we can make the following two substitutions:

$$\frac{\delta L}{\delta \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \frac{\delta L}{\delta \phi} = -\sum_n \frac{\partial}{\partial x_n}\frac{\partial \mathcal{L}}{\partial[\frac{\partial \phi}{\partial x_n}]} + \frac{\partial \mathcal{L}}{\partial \phi}$$

Plugging this in and using the Einstein summation convention to simplify our notation (note that here X represents the 4-vector $X = (t, \vec{x})$):

$$\frac{\partial}{\partial X^\mu}\frac{\partial \mathcal{L}}{\partial[\frac{\partial \phi}{\partial X^\mu}]} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

We will also need to justify our canonical commutators. The standard canonical commutators for some discrete collection of generalized coordinates $\{\phi_n\}$ are (remember that L represents the Lagrangian not the Lagrangian density):

$$[\phi_n, \frac{\partial L}{\partial \dot{\phi}_m}] = i\delta_{nm}$$

To generalize this to a continuous field we will need to reinterpret the partial derivative of the Lagrangian. To do this, we make the following definition of the partial derivative of a field:

$$\frac{\partial}{\partial \phi(\vec{x})} f(\phi(\vec{y})) = \begin{cases} f'(\phi(\vec{x})) & \vec{x} = \vec{y} \\ 0 & \vec{x} \neq \vec{y} \end{cases}$$

Let's use this, and the definition of L in terms of $\mathcal{L}$ to expand the following expression. What we are looking for is a form for $[\phi(x), \pi(y)]$:

$$[\phi(\vec{x}), \frac{\partial L}{\partial \dot{\phi}(\vec{x})}] = i$$

$$\int [\phi(\vec{x}), \frac{\partial \mathcal{L}(\vec{y})}{\partial \dot{\phi}(\vec{x})}] d^3x d^3y = i$$

$\frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})}$ is only non-zero when $x = y$ so we can substitute x for y in that part of the expression yielding:

$$\int [\phi(\vec{x}), \frac{\partial \mathcal{L}(\vec{y})}{\partial \dot{\phi}(\vec{y})}] d^3x d^3y = i$$

From here we use the standard definition for the conjugate momentum of a field:

$$\pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(\vec{x})$$

Plugging this in we get:

$$\int [\phi(\vec{x}), \pi(\vec{y})] d^3x d^3y = i$$

However above we just said that the term $\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ is only non-zero when $\vec{x} = \vec{y}$ so we have:

$$\vec{x} \neq \vec{y} \quad [\phi(\vec{x}), \pi(\vec{y})] = 0$$

The only function that satisfies these two conditions is:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y})$$

Now that we have that out of the way we can use our equations on the Klein Gordon field. We plug our definition of the action into the Euler Lagrange equations:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \qquad \frac{\partial}{\partial X^\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial X^\mu}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$(\partial_\mu \partial^\mu + m^2)\phi(\vec{x}) = 0$$

As expected we get the Klein Gordon equation. To solve this equation we write the field in terms of it's Fourier Transform (note that I've dropped some factors of 2π, had I included them they would have been divided out later in the analysis):

$$\phi(\vec{x},t) = \int \hat{\phi}(\vec{p},t)e^{i\vec{p}\cdot\vec{x}}d^3p$$

We can then plug this into the Klein Gordon equation and solve:

$$0 = (\partial_\mu\partial^\mu + m^2)\phi(\vec{x}) = (\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\phi(\vec{x})$$

$$= \int \ddot{\hat{\phi}}(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} - \hat{\phi}(\vec{p},t)\nabla^2 e^{i\vec{p}\cdot\vec{x}} + m^2\hat{\phi}(\vec{p},t)e^{i\vec{p}\cdot\vec{x}}d^3p$$

$$= \int [\ddot{\hat{\phi}}(\vec{p},t) + |\vec{p}|^2\hat{\phi}(\vec{p},t) + m^2\hat{\phi}(\vec{p},t)]e^{i\vec{p}\cdot\vec{x}}d^3p$$

This equation must hold for all values of $\vec{x}$ so we can drop the integral and evaluate for each momentum state individually:

$$\ddot{\hat{\phi}}(\vec{p},t) + |\vec{p}|^2\hat{\phi}(\vec{p},t) + m^2\hat{\phi}(\vec{p},t) = 0$$

$$\ddot{\hat{\phi}}(\vec{p},t) = -(|\vec{p}|^2 + m^2)\hat{\phi}(\vec{p},t)$$

This is just the equation for a simple harmonic oscillator with frequency $\sqrt{|\vec{p}|^2 + m^2}$ which has solutions:

$$\hat{\phi}(\vec{p},t) = A(\vec{p})e^{it\sqrt{|\vec{p}|^2+m^2}} + B(\vec{p})e^{-it\sqrt{|\vec{p}|^2+m^2}}$$

We will simplify this expression by calling the frequency E. Thus:

$$\hat{\phi}(\vec{p},t) = A(\vec{p})e^{itE} + B(\vec{p})e^{-itE}$$

We can then plug this back into our original form for the field operator:

$$\phi(\vec{x},t) = \int A(\vec{p})e^{iEt+i\vec{p}\cdot\vec{x}} + B(\vec{p})e^{-iEt+i\vec{p}\cdot\vec{x}}d^3p$$

We can then do a little rearranging, namely reversing the order over which we integrate the term involving $B(\vec{p})$:

$$\phi(\vec{x},t) = \int A(\vec{p})e^{iEt+i\vec{p}\cdot\vec{x}} + B(\vec{p})e^{-iEt+i\vec{p}\cdot\vec{x}}d^3p$$

$$= \int A(\vec{p}) e^{iEt + i\vec{p}\cdot\vec{x}} d^3p + \int B(\vec{p}) e^{-iEt + i\vec{p}\cdot\vec{x}} d^3p$$

$$= \int A(\vec{p}) e^{iEt + i\vec{p}\cdot\vec{x}} d^3p + \int B(-\vec{p}) e^{-iEt - i\vec{p}\cdot\vec{x}} d^3p$$

$$= \int A(\vec{p}) e^{iEt + i\vec{p}\cdot\vec{x}} + B(-\vec{p}) e^{-iEt - i\vec{p}\cdot\vec{x}} d^3p$$

To simplify our notation we can recognize that the term in the exponent is just the product of two 4-vectors, X and a new vector P defined $P = (E, \vec{p})$:

$$\phi(X) = \int A(\vec{p}) e^{iP_\mu X^\mu} + B(-\vec{p}) e^{-iP_\mu X^\mu} d^3p$$

Next we explicitly enforce the fact that the field $\phi(X)$ is a hermitian operator:

$$\phi^\dagger(X) = \phi(X)$$

$$\int A(\vec{p}) e^{iP_\mu X^\mu} + B(-\vec{p}) e^{-iP_\mu X^\mu} d^3p = \int A^\dagger(\vec{p}) e^{iP_\mu X^\mu} + B^\dagger(-\vec{p}) e^{-iP_\mu X^\mu} d^3p$$

Thus we can see:

$$A^\dagger(\vec{p}) = B(-\vec{p})$$

We can then plug this into our expression for the field to write it entirely in terms of the set of operators $A(\vec{p})$. In this form it is explicitly hermitian as it is written in as the sum of an operator and it's hermitian conjugate. Note that we are still integrating over d^3p and not over d^4P as the 0th component of the momentum 4-vector still depends on the others:

$$\phi(X) = \int A(\vec{p}) e^{iP_\mu X^\mu} + A^\dagger(\vec{p}) e^{-iP_\mu X^\mu} d^3p$$

Now that we have a form for $\phi(X)$ in terms of our new operator $A(\vec{p})$ we need to find the commutators of A. We are doing this because we are trying to find the particle creation and annihilation operators which should be hidden somewhere in the field.

We will do this by enforcing the canonical commutators while making two simplifying assumptions, namely that the only non-zero commutators those with an A and an $A^\dagger$ of the same momentum. This is of course only a guess and it will eventually be justified by the form we find for the energy. If the energy didn't reproduce the time dependence we'd found for A then we'd know that our assumptions had been inconsistent. As we will soon see they are not.

First we write out the canonical commutators for the field in position space:

$$[\phi(\vec{x}), \dot{\phi}(\vec{y})] = i\delta^3(\vec{x} - \vec{y})$$

We then plug in our form for the field in terms of A and simplify using the assumptions mentioned above:

$$i\delta^3(\vec{x} - \vec{y}) = \int \int 2iE[A(\vec{p}), A^\dagger(\vec{q})]e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} d^3p d^3q$$

This expression must only depend on $\vec{x} - \vec{y}$ and no other combination so this means $[A(\vec{p}), A^\dagger(\vec{q})]$ must be 0 whenever $\vec{p} \neq \vec{q}$. We can thus make the following definition:

$$[A(\vec{p}), A^\dagger(\vec{q})] = F(\vec{p})\delta^3(\vec{p} - \vec{q})$$

To find a form for F we plug it back into the above expression. Note that I've canceled the i on both sides of the expression:

$$\delta^3(\vec{x} - \vec{y}) = \int 2EF(\vec{p})e^{i\vec{p}\cdot(\vec{x} - \vec{y})} d^3p$$

In order for this integral to evaluate to the dirac delta our commutator must be:

$$[A(\vec{p}), A^\dagger(\vec{q})] = \frac{1}{16\pi^3 E}$$

We are looking for raising and lowering operators which have commutator 1. To this end we do the following manipulation:

$$[(4\sqrt{\pi^3 E}A(\vec{p})), (4\sqrt{\pi^3 E}A(\vec{p}))^\dagger] = 1$$

We can call the new operator $\psi(\vec{p})$ and rewrite the field in terms of it:

$$\psi(\vec{p}) = 4\sqrt{\pi^3 E}A(\vec{p})$$

$$\phi(X) = \int \left[\psi(\vec{p})e^{i(P_\mu X^\mu)} + \psi^\dagger(\vec{p})e^{-i(P_\mu X^\mu)}\right] \frac{d^3p}{4\sqrt{\pi^3 E}}$$

Though the ψs have the right commutators for creation and annihilation operators we are not done. We need to show that the operator they raise and lower is the energy and that they raise/lower by the right amount to be adding/removing particles. To do this, we will write out the energy in terms of ψ by using our definition of ϕ. The calculation is very long and not particularly enlightening so I will only show the end result here:

$$\mathcal{H} = \int E\psi^\dagger(\vec{p})\psi(\vec{p})d^3p$$

We have one more thing to do before we're done. We need to justify our assumptions from earlier about the form for the commutators of A and thus those of ψ. To do this, we can calculate the first derivatives of the ψs using Heisenbergs formula for time evolution. If it agrees with what we already have, than our assumptions were consistent.

$$\dot{\psi}(\vec{p}) = i[\mathcal{H}, \psi(\vec{p})] = -iE\psi(\vec{p})$$

This is the exact same equation we derived earlier for the time dependence of A. Thus the assumptions we made earlier related to the commutators of the ψs were correct.

Finally we can analyze the form we've found for the energy. The term in the middle is the number operator whose eigenvalues are raised and lowered by the ψs. We can interpret this as the number of particles of momentum $\vec{p}$. This is then multiplied by the energy of each of the particles. This is exactly the energy we would expect for a collection of non-interacting relativistic particles!

This is one of my favorite problems I've solved. It blows my mind that it's possible to start with the Klein Gordon equation, an equation for fields, and end up with a Hamiltonian that describes particles

Deriving the Canonical Quantization Approach from the Path Integral

Max Orton

Spring 2024

My work with the path integral was another attempt to make quantum mechanics consistent with special relativity. It took me several attempts to find its correct form as I began this work knowing only that the amplitude involved a sum over paths weighted by a complex exponential of the action.

Our goal is to start with the basic principles of the path integral approach and use them to derive canonical quantization. We will be studying an n particle system with positions and momenta $\{q_i, p_i\}$. Each state is characterized by a wave function $\psi(\{q_i\})$. For our purposes, it is a function only of the $\{q_i\}$ as we will work only in position space. The amplitude that state ψ_I will transition to state ψ_F over elapsed time $t_I \to t_F$ is given by:

$$\langle \psi_F, t_F | \psi_I, t_I \rangle \propto \int \overline{\psi}_F(\{q_i(t_F)\})\psi_I(\{q_i(t_I)\})e^{iS[\{q_i\}]}\mathcal{D}q_i$$

Note that for our purposes, $S[\{q_i\}]$ is the action that exists between t_I and t_F.

$$S[\{q_i\}] = \int_{t_I}^{t_F} \mathcal{L}(\{q_i(t)\})dt$$

First, we will find the form of the momentum operator. We can do this by multiplying it by two vectors, ψ_1 and ψ_2 on the right and left respectively. We have to shift ψ_2 forward by an infinitesimal time shift ϵ because momentum doesn't make sense in the path integral without some amount of elapsed time.

Later, we'll let it tend to zero.

$$\langle\psi_2, t|\, p_i \,|\psi_1, t\rangle \propto \lim_{\epsilon\to 0}\int \overline{\psi}_2(\{q_i(t{+}\epsilon)\})\psi_1(\{q_i(t)\})p_i(t)\exp\left\{i\int_t^{t+\epsilon}\mathcal{L}(\tau)d\tau\right\}\mathcal{D}q_i$$

We can solve this integral by discritizing time. We will break it down into steps of size epsilon which effectively makes our path a straight line from the initial point to the final point. Amazingly, this is enough resolution to find the momentum operator. With this restriction on our path $q(t) = q$, $q(t + \epsilon) = q + \epsilon\dot{q}$ and $S = \epsilon\mathcal{L}$. We will also use the fact that the cannonical momentum to q_i is $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$. Plugging all this in, we get:

$$\langle\psi_2, t|\, p_i \,|\psi_1, t\rangle \propto \lim_{\epsilon\to 0}\int \overline{\psi}_2(\{q_i + \epsilon\dot{q}_i\})\psi_1(\{q_i(t)\})p_i(t)\exp\{i\epsilon\mathcal{L}(t)\}\, dq_i d\dot{q}_i$$

We can rewrite p_i as coming from a derivative of the exponential,

$$\langle\psi_2, t|\, p_i \,|\psi_1, t\rangle \propto \lim_{\epsilon\to 0}\frac{-i}{\epsilon}\int \overline{\psi}_2(\{q_i + \epsilon\dot{q}_i\})\psi_1(\{q_i(t)\})\frac{\partial}{\partial\dot{q}_i}\exp\{i\epsilon\mathcal{L}(t)\}\, dq_i d\dot{q}_i$$

integrate by parts and do the chain rule,

$$\langle\psi_2, t|\, p_i \,|\psi_1, t\rangle \propto \lim_{\epsilon\to 0} i\int \frac{\partial}{\partial q_i}\left(\overline{\psi}_2(\{q_i + \epsilon\dot{q}_i\})\right)\psi_1(\{q_i(t)\})\exp\{i\epsilon\mathcal{L}(t)\}\, dq_i d\dot{q}_i$$

let epsilon go to zero,

$$\langle\psi_2|\, p_i \,|\psi_1\rangle \propto i\int \frac{\partial}{\partial q_i}\left(\overline{\psi}_2(\{q_i\})\right)\psi_1(\{q_i(t)\})dq_i$$

and integrate by parts again.

$$\langle\psi_2|\, p_i \,|\psi_1\rangle \propto \int \overline{\psi}_2(\{q_i\})\left[-i\frac{\partial}{\partial q_i}\right]\psi_1(\{q_i(t)\})dq_i$$

Up to a constant of proportionality, we've found the momentum operator!

Our next goal is to show that states, evolve by repeated application of $(1 - i\epsilon\mathcal{H})$ where $\mathcal{H}$ is the Hamiltonian. To do this, we will take the inner product of two states separated by a small time step δt. In this case, $\delta t = 2\epsilon$, where epsilon is the size of the discretization we will use to solve the path integral. Finding the energy operator requires this slightly longer path. We will be trying to show:

$$\langle\psi_2, t+2\epsilon|\psi_1, t\rangle = \langle\psi_2, t+\epsilon|\left(1 - i\epsilon\mathcal{H}\right)|\psi_1, t\rangle$$

Our first step is to write out the inner product as a path integral.

$$\langle\psi_2, t+2\epsilon|\psi_1, t\rangle \propto \int \overline{\psi}_2(\{q_i(t+2\epsilon)\})\psi_1(\{q_i(t)\}) \exp\left(i\int_t^{t+2\epsilon} \mathcal{L}(\tau)d\tau\right)\mathcal{D}q$$

Again, we will discretize time into steps of size $\epsilon = \frac{1}{2}\delta t$. We than make the following substitutions:

1: $q_i(t+n\epsilon) = q_n^i$
1: $\dot{q}_i(t+n\epsilon) = \dot{q}_{n,n+1}^i$
2: $\mathcal{L}(t+n\epsilon) = \mathcal{L}_{n,n+1}$
3: $S = \epsilon\mathcal{L}_{01} + \epsilon\mathcal{L}_{12}$

Now we can rewrite our integral in terms of these new variables:

$$\langle\psi_2, t+2\epsilon|\psi_1, t\rangle \propto \int \overline{\psi}_2(\{q_1^i + \epsilon\dot{q}_{12}^i\})\psi_1(\{q_0^i\}) \exp\left(i\epsilon\mathcal{L}_{01} + i\epsilon\mathcal{L}_{12}\right) dq_0^i\, d\dot{q}_{01}^i\, dq_{12}^i$$

Now we expand to first order in epsilon. We have to do this carefully because we only want to expand around the second time step. Thus, we will leave the $\exp(i\epsilon\mathcal{L}_{01})$ untouched. The first order term is:

$$\epsilon\left(\int\left[i\overline{\psi}_2(\{q_1^i\})\mathcal{L}_{12} + \sum_i\left(\frac{\partial\overline{\psi_2}}{\partial q_i}\dot{q}_{12i}\right)\right]\psi_1(\{q_0^i\})\exp\left(i\epsilon\mathcal{L}_{01}\right) dq_0^i\, d\dot{q}_{01}^i\, dq_{12}^i\right)$$

We can break this into two separate integrals and evaluate them individually before adding them back together. By far the simpler of the two is the first term in the brackets above (the one multiplied by the Lagrangian between times one and two).

$$i\epsilon\int \mathcal{L}_{12}\overline{\psi}_2(\{q_1^i\})\psi_1(\{q_0^i\}))\exp(i\epsilon\mathcal{L})dq_0^i\, d\dot{q}_{01}^i\, dq_{12}^i$$

This term doesn't require much manipulation. All we need to do is rewrite it to have the same coefficient as the Hamiltonian in the time evolution operator. This gives us the negative of the Lagrangian which is, indeed, a term in the Hamiltonian!

$$-i\epsilon \int (-\mathcal{L}_{12})\overline{\psi}_2(\{q_1^i\})\psi_1(\{q_0^i\}))\exp(i\epsilon\mathcal{L}_{01})dq_0^i\, d\dot{q}_{01}^i\, dq_{12}^i$$

The other term will require a little more manipulation. This will amount to the inverse of what we did with the momentum operator. There, we took a factor of momentum in the integrand and replaced it with a derivative. Here, we will take a derivative and replace it with a factor of the momentum in the integrand. We do the following steps:

$$\epsilon \sum_i \int \dot{q}_{12i}\frac{\partial}{\partial q_{0i}}\overline{\psi}_2(\{q_{0i}+\epsilon\dot{q}_{01i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

We can replace the derivative with respect to q_{0i} with a derivative with respect to $\dot{q}_{01i}$, which cancels the ϵ out front.

$$\sum_i \int \dot{q}_{12i}\frac{\partial}{\partial \dot{q}_{01i}}\overline{\psi}_2(\{q_{0i}+\epsilon\dot{q}_{01i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

Then, we integrate by parts, shifting the derivative from $\overline{\psi}_2$ onto the rest of the integrand (at the cost of a negative sign). The only thing besides $\overline{\psi}_2$ that depends on $\dot{q}_{01i}$ is $\mathcal{L}_{01}$. This gives us:

$$-\sum_i \int \dot{q}_{12i}\overline{\psi}_2(\{q_{0i}+\epsilon\dot{q}_{01i}\})\psi_1(\{q_{0i}\})\frac{\partial}{\partial \dot{q}_{01i}}\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

Evaluating this derivative, we pull down a factor of i, a factor of ϵ, and a derivative of $\mathcal{L}_{01}$ with respect to $\dot{q}_{01i}$, which we recognize as the conjugate momentum p_{01i}!

$$\int \left(-i\epsilon \sum_i \dot{q}_{12i}p_{01i}\right)\overline{\psi}_2(\{q_{1i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

Adding these back together, we have an expression for our original inner product (where k is a constant of proportionality that normalizes our path integral).

$$\langle\psi_2, t+\epsilon|\psi_1, t\rangle - ik\epsilon \int \left(\sum_i \dot{q}_{12i}p_{01i} - \mathcal{L}_{12}\right)\overline{\psi}_2(\{q_{1i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

But... the expression in parenthesis is just the Hamiltonian! Rewriting this all in terms of inner products, we've derived the expression for the evolution of a quantum state in the *canonical quantization* approach only using the *path integral*!

$$\langle \psi_2, t + 2\epsilon | \psi_1, t \rangle = \langle \psi_2, t + \epsilon | \left(1 - i\epsilon\mathcal{H}\right) | \psi_1, t \rangle$$

There is a lot I plan to do with the path integral approach in the future. For instance, I have hopes to recast the work I've done in quantum field theory in terms of the path integral. The way I've written it so far explicitly singles out time. I hope to eventually remove this, perhaps by creating a way to measure and restrict the amount of information stored in a given state.

Solutions to the Wave Equation

Max Orton

Sophomore and Junior Year: 2022 through 2023

The wave equation is one of the earliest projects I tried to figure out. I started investigating it during my Freshman year as a stepping stone toward Maxwell's Equations. This write-up describes a model I created that involves breaking an initial condition into components, each defined only along a single axis. It is the simplest general solution I've created so far.

We will be investigating and finding solutions to the wave equation:

$$\Box\,\phi = 0 \quad \text{or} \quad \frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$$

This problem was a priority of mine, primarily because it was fun, but also because the wave equation turns out to be equivalent to a surprising number of other interesting PDEs. A good example of this is the Klein Gordon equation. Say we have a field ϕ that satisfies the wave equation in N dimensional space (*space* not spacetime).

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \cdots + \frac{\partial^2 \phi}{\partial x_N^2}$$

We can invent a new field ψ that depends only on N-1 of the coordinates. It is defined:

$$\phi(x_1, x_2, \cdots x_N) = \psi(x_2, \cdots x_N) e^{imx_1}$$

If we plug this into our equation for ϕ, we can get an expression for ψ.

$$\frac{\partial^2 \psi}{\partial t^2} e^{imx_1} = \left(\frac{\partial^2}{\partial x_1^2} \psi e^{imx_1} \right) + \frac{\partial^2 \psi}{\partial x_2^2} e^{imx_1} + \cdots + \frac{\partial^2 \psi}{\partial x_N^2} e^{imx_1}$$

Expanding this, we find that ψ follows the Klein Gordon Equation.

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x_2^2} + \cdots + \frac{\partial^2\psi}{\partial x_N^2} - m^2\psi$$

Before we solve the entire wave equation, we can start with the one dimensional version. We see that it is:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\phi = 0$$

We can factor this expression into two terms. It is worth noting that this is not the way I originally solved this equation. When I first solved it, I Fourier Transformed it in space, but not in time, which made it a harmonic oscillator. I solved that equation, then reversed the Fourier Transform. As it is, we will use a simpler method I discovered later.

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)\phi = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi = 0$$

Only one of the two factors, when operated on ϕ, needs to yield zero for this equation to be satisfied. This gives us two equations, and their solutions can be written in terms of two arbitrary functions: $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\phi_L \to \phi_L(x,t) = f(x+ct)$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi_R \to \phi_R(x,t) = g(x-ct)$$

Because of linearity, ϕ can be written as a sum of these two solutions.

$$\phi(x,t) = f(x+ct) + g(x-ct)$$

Now we can move on to the N dimensional wave equation. The obvious solution is to write it as a Fourier Transform. For a real field, this version of the solution would have the form:

$$\phi(\vec{x},t) = \int \alpha(\vec{k})e^{i(\vec{k}\cdot\vec{x}+t\|\vec{k}\|)} + \overline{\alpha}(\vec{k})e^{-i(\vec{k}\cdot\vec{x}+t\|\vec{k}\|)}d^N k$$

This expression can be very useful. I've used it extensively when working with quantum fields, but it is not the most satisfying answer. After all, it requires N different integrals to evaluate the expression, not to mention at least N more to set the initial conditions.

We are going to notice something else. If a function depends only on one of the x's, it must satisfy the one dimensional wave equation as all the other derivatives would be zero. Likewise, since the wave equation is rotationaly symmetric, a function that depends only on any unit vector dotted with position must follow the one dimensional wave equation with respect to a directional derivative along that unit vector. Any linear combination of these vectors would also need to satisfy the wave equation. This gives us a form for ϕ: basically just a superposition of plane waves f_n moving along a collection of corresponding unit vectors α_n.

$$\phi(\vec{x}, t) = \sum_n f_n(\vec{x} \cdot \hat{\alpha}_n - ct)$$

The question now becomes, how do we find the set of f_n given an initial condition? Basically we want an object $\mathfrak{F}$ defined.

$$\text{if} \quad g(\hat{\alpha}, r) = \mathfrak{F}[f(\vec{x})](\hat{\alpha}, r) \quad \text{then} \quad f(\vec{x}) = \int g(\hat{\alpha}, \vec{x} \cdot \hat{\alpha}) d^{N-1}\alpha$$

I had a small hint with this one. One of the SoME3 submissions was about "string art" (creating pictures by connecting strings at the edges of a circle), and involved solving a problem similar to this one. The person in the video solved it by just brute-forcing the problem in a way that could not be done symbolically. However, at the very end, he mentioned that he now had a new algorithm involving the Fourier Transform. He did not explain how it worked (honestly, I was glad, as I wanted to figure it out myself), but it gave me a place to start as I knew the Fourier Transform would be involved.

Since we know the solution will somehow involve the Fourier Transform, we can begin by writing f in terms of its Fourier Transform, then manipulate it to try and get it into a form similar to the equation above.

$$f(\vec{x}) = (2\pi)^{-\frac{N}{2}} \int \hat{f}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} d^3k$$

$$f(\vec{x}) = \int (2\pi)^{-\frac{N-1}{2}} \frac{1}{\sqrt{2\pi}} \int \left[\hat{f}(\hat{k}, \|k\|) \exp\left(-i\|k\|(\hat{k} \cdot \vec{x})\right)\right] \|k\|^{N-1} d\|k\| d^{N-1}\hat{k}$$

This has the form of an integral over a unit vector $\hat{k}$ of a function that depends on $\vec{x}$ only through $\vec{x} \cdot \hat{k}$. We can use this to write out an expression for $\mathfrak{F}$!

$$\mathfrak{F}[f(\vec{x})] = \mathcal{F}^{-1}_{\|\vec{k}\|}[\mathcal{F}_x[f(\vec{x})](\vec{k})](\hat{k}, r)$$

This means all we have to do to solve the wave equation is evaluate $\mathfrak{F}$ on our initial condition to get the f_n, then add them all back together in the right way to get ϕ. A few moments of thought will show that we actually have the same total number of integrals to perform in this process as in the original Fourier Transform method. That said, this solution still does have an advantage. We've shifted one of our 2N integrals from the evaluation to setting the initial conditions. We will often want to evaluate a solution at multiple points in time to get a feel for how it evolves. This solution saves us one integral for each evaluation, save the first one!

My quest to solve this problem has taught me more than I could ever have imagined at the start. When I began, I barely understood the difference between an ODE and a PDE. Now I have homemade methods for solving both. It sometimes felt like I was approaching this with a strategy that valued quantity over quality. I invented dozens of methods, hoping that one might just apply to the wave equation. This process has left me with tools to solve all kinds of other PDEs. I don't consider this project complete. Though this decomposition method is the simplest I've found so far, I'm hopeful that a simpler solution exists. A lot of my more recent work on the wave equation and other PDEs revolves around creating new number systems in which they can be more easily solved: trying to generalize the fact that an arbitrary function of a complex number reproduces Laplace's Equation in 2D to more equations and corresponding number systems. I've used this to find exact solutions for all 2D PDEs where the total number of derivatives on each term is constant. I'm also exploring ways to extend this method to more complex PDEs and in more dimensions.

My Notebook Excerpts

$\vec{F} = \dot{p} = \vec{B} \times \vec{V}$

~~$\mathcal{L} = -m \frac{d\tau}{dt} + qA_\mu \frac{dx^\mu}{d\tau}$~~

halfway through driving the Lorenze force law I remembered google

$\frac{d}{dt}\left(\frac{\dot{x} \cdot m}{\sqrt{1-|\dot{x}|^2}}\right) = q(\vec{E} + \dot{x} \times \vec{B})$

$\dot{p} = q(\vec{E} + \vec{V} \times \vec{B})$ $\quad p = \frac{vm}{\sqrt{1-\frac{v^2}{c^2}}}$

$V^2\left(1 + \frac{p^2}{c^2}\right) = p^2$

$\vec{V} = \frac{\vec{p}}{\sqrt{1+\frac{p^2}{c^2}}}$

$\dot{\vec{p}} = q\left(\vec{E}^{\,0} + \frac{\vec{p} \times \vec{B}}{\sqrt{1+p^2}}\right)$

$E = \sqrt{p^2 + m^2}$

$\dot{p} = \frac{q}{\sqrt{1+\frac{p^2}{c^2}}}(p \times B)$

$B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}$, $p = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix}$ $\quad p \times B = \begin{bmatrix} -bp_y \\ bp_x \\ 0 \end{bmatrix}$

$p_x = p_0 \cos\left(\frac{qbt}{\sqrt{m^2 + \frac{p_0^2}{c^2}}}\right)$

$p_y = p_0 \sin\left(\frac{qbt}{\sqrt{m^2 + \frac{p_0^2}{c^2}}}\right)$

$E^2 = p^2c^2 + m^2c^4$

$\frac{E^2 - m^2c^4}{c^2} = p^2$

$\frac{E^2}{c^4} - m^2 = \frac{p^2}{c^2}$

$p_0 = \frac{1}{c}\sqrt{E^2 - m^2c^4}$

$p_0\omega \cos(\omega t) = \frac{qb}{\sqrt{1+\frac{p_0^2}{c^2}}} p_0 \cos(\omega t)$

$p_x = p_0 \cos\left(\frac{qcbt}{\sqrt{c^2 + p_0^2}}\right)$ $\quad p_y = p_0 \sin\left(\frac{qcbt}{\sqrt{c^2+p_0^2}}\right)$

$\dot{p}_x = \frac{-qb}{\sqrt{m^2 + \frac{p_x^2 + p_y^2}{c^2}}} p_y$

$\dot{p}_y = \frac{qb}{\sqrt{m^2 + \frac{p_x^2+p_y^2}{c^2}}} p_x$

some important ODE's

everyone feels weird without the (admitedly redundant) "0"

$p_x = p_0 \cos(\omega t)$ $\quad p_y = p_0 \sin(\omega t)$

$p_0\omega \sin(\omega t) = \frac{+qb}{\sqrt{m^2 \frac{p_0^2}{c^2}}}(p_0 \sin(\omega t))$

$\vec{V} = \frac{\vec{p}}{\sqrt{1+\frac{p^2}{c^2}}}$

$p^2 = v^2\left(m^2 + \frac{p^2}{c^2}\right)$

$\frac{p}{\sqrt{m^2 + \frac{p^2}{c^2}}} = \frac{\frac{p}{m}}{\sqrt{1+\frac{(\frac{p}{m})^2}{c^2}}} = v$ $\quad \frac{pc}{\sqrt{m^2c^2 + p^2}}$

$\omega = \frac{qb}{\sqrt{m^2 \frac{p_0^2}{c^2}}}$

$\frac{qb}{\sqrt{m^2 + \frac{p_0^2}{c^2}}}$

$\frac{qb}{\sqrt{m^2 + \frac{E^2}{c^4} - m^2}}$

$\vec{V} = \frac{\vec{p}}{\sqrt{m^2 + \frac{p^2}{c^2}}} =$

$v_0 = \frac{1}{c}\sqrt{E^2 - m^2c^4}\left(\frac{c^2}{E}\right) = c\sqrt{1 - \frac{m^2c^4}{E^2}} = v_0$

$v_x = p_0\left(\frac{1}{\sqrt{m^2 + \frac{p_0^2}{c^2}}}\right)\cos(\omega t) = v_0 \cos(\omega t)$

$v_y = p_0\left(\frac{1}{\sqrt{m^2 + \frac{p_0^2}{c^2}}}\right)\sin(\omega t) = v_0 \sin(\omega t)$

$x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0$

$y(t) = -\frac{v_0}{\omega}\cos(\omega t) + y_0$

$p_x = p_0 \cos\left(\frac{qbc^2}{E}t\right)$

$p_y = p_0 \sin\left(\frac{qbc^2}{E}t\right)$

E_2 E_1 x_2 x_1

at this point I gave up on doing everything by hand and started useing mathematica

$v_0 = c\sqrt{1 - \frac{m^2c^4}{E^2}}$

$\omega = \frac{qbc^2}{E}$

DON'T FORGET THESE DEFINITIONS!

$x_1(t) = \frac{v_0}{\omega}\sin(\omega t)$

$y_1(t) = -\frac{v_0}{\omega}\cos(\omega t) + \frac{v_0}{\omega}$

$x_2(t) = \frac{v_0}{\omega}\sin(\omega t)$

$y_2(t) = -\frac{v_0}{\omega}\cos(\omega t) + \frac{v_0}{\omega}$

distance between particles

~~$d(t) = \sqrt{\frac{v_0^2}{\omega^2}[\sin\omega_1 t - \sin\omega_2 t]^2 + \frac{v_0^2}{\omega^2}[\cos(\omega_1 t) - \cos(\omega_2 t) - \frac{v_1^0}{\omega_1} + \frac{v_2^0}{\omega_2}]^2}$~~

$d(t) = \sqrt{\left(\frac{v_1^0}{\omega_1}\sin(\omega_1 t) - \frac{v_2^0}{\omega_2}\sin(\omega_2 t)\right)^2 + \left(\frac{v_1^0}{\omega_1}[\cos\omega_1 t - 1] - \frac{v_2^0}{\omega_2}[\cos\omega_2 t - 1]\right)^2}$

~~$d(t) \approx d(0) + \dot{d}(0)t$~~ ← cause we're working with times that are tiny compared with all other variables (in the) problem

$d(t) \approx \dot{d}(0)t + \frac{t^2}{2}\ddot{d}(0)$

unfortunately we need more terms... $\dot{d}$ is 0 and $\dot{d}(0)$ isn't usefull not depending on magnetic field

$\partial_t \vec{\psi} = \hat{H}\vec{\psi}$

$\vec{\psi}(x,t) = \int \hat{K}(x,x_0,t)\psi(x_0,0)\,dx_0$

~~ ~~ $\dot{\hat{x}} = [\hat{x},\hat{H}]$

$\hat{x}(t)\hat{K}(x,x_0,t) = x_0\hat{K}(x,x_0,t)$

method for solving linear PDE's

Definition of inner product

$\left\langle \text{final state} \middle| \text{initial state} \right\rangle = (\text{amplitude})$

I've had this backward for soooooo long...

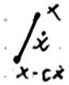
$[x,P] = i\hbar$

$P \Rightarrow -i\hbar\frac{\partial}{\partial x}$

$\int f'g + g'f\,dx = fg$

$\int f'g\,dx = -\int g'f\,dx$

$\langle\psi_1|P|\psi_2\rangle = \int \psi_1^*(x)\left[-i\hbar\frac{\partial}{\partial x}\psi_2(x)\right]dx$



$\int \psi_1^*(x+\epsilon\dot{x})\psi_2(x)e^{\frac{iS}{\hbar}}\,\mathcal{D}x$

$\iint \psi_1^*(x+\epsilon\dot{x})\psi_2(x)\frac{\partial x}{\partial \dot{x}}e^{\frac{i\epsilon}{\hbar}\mathcal{L}(x-\epsilon\dot{x},\dot{x})}\,dx\,d\dot{x}$

$\iint \psi_1^*(x+\epsilon\dot{x})\psi_2(x)\frac{\partial \mathcal{L}}{\partial \dot{x}}e^{\frac{i\epsilon}{\hbar}\mathcal{L}(x,\dot{x})}\,dx\,d\dot{x}$

$\frac{-i\hbar}{\epsilon}\iint \psi_1^*(x+\epsilon\dot{x})\psi_2(x)\frac{\partial}{\partial \dot{x}}e^{\frac{i\epsilon}{\hbar}\mathcal{L}(x,\dot{x})}\,dx\,d\dot{x}$

~~$+i\hbar\iint \psi_1^{*\prime}(x$~~

$i\hbar\int \psi_1^{*\prime}(x)\psi_2(x)\,dx$

$\int -i\hbar\psi_1^*(x)\psi_2'(x)\,dx$ ✓

HAH!
It WORKS!!!

$\langle\psi_2|\psi_1\rangle = \int \psi_2^*(x(t))\psi_1(x(0))\,e^{\frac{iS[x]}{\hbar}}\,\mathcal{D}x$

1D fluid flow:

$$\partial_t v = -\partial_x p$$
$$\partial_t p = -\partial_x v$$

$|s\rangle = \begin{bmatrix} v \\ p \end{bmatrix}$ $\quad \mathbb{I} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\partial_t |s\rangle = -\partial_x \mathbb{I} |s\rangle$

$\partial_t |s\rangle = -\mathbb{I}\, \partial_x |s\rangle$

$\partial_t |\hat{s}\rangle = 2i\pi k \mathbb{I} |\hat{s}\rangle$

$|\hat{s}\rangle = e^{2i\pi k \mathbb{I}} |\hat{s}_0\rangle$

$|s(x,t)\rangle = F^{-1}[e^{2i\pi k \mathbb{I} t}] * |s(x,0)\rangle$

$e^{2i\pi k \mathbb{I} t} = \frac{1}{2}\begin{bmatrix} e^{2i\pi k t} - e^{-2i\pi k t} & e^{2i\pi k t} + e^{-2i\pi k t} \\ e^{2i\pi k t} + e^{-2i\pi k t} & e^{2i\pi k t} - e^{-2i\pi k t} \end{bmatrix}$

$e^{\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} e^{a} + e^{-a} & e^{a} + e^{-a} \\ e^{a} + e^{-a} & e^{a} - e^{-a} \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}^n = \begin{cases} \begin{bmatrix} 0 & a^n \\ a^n & 0 \end{bmatrix} & \frac{1}{2} n \in \mathbb{Z} \\ \begin{bmatrix} a^n & 0 \\ 0 & a^n \end{bmatrix} & \frac{1}{2} n \notin \mathbb{Z} \end{cases}$

$\frac{1}{2}\begin{bmatrix} \delta(x+t) - \delta(x-t) & \delta(x+t) + \delta(x-t) \\ \delta(x+t) + \delta(x-t) & \delta(x+t) - \delta(x-t) \end{bmatrix} * \begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(v_0(x+t) - v_0(x-t)) + \frac{1}{2}(p_0(x+t) + p_0(x-t)) \\ \frac{1}{2}(v_0(x+t) + v_0(x-t)) + \frac{1}{2}(p_0(x+t) - p_0(x-t)) \end{bmatrix}$

$$v(x,t) = \tfrac{1}{2}(v(x+t,0) + v(x-t,0) + p(x+t,0) - p(x-t,0))$$
$$p(x,t) = \tfrac{1}{2}(p(x+t,0) + p(x-t,0) + v(x+t,0) - v(x-t,0))$$

complete solution for 1D fluid flow ↗

$\ddot{f} = af + C$

$\dot{f} - af = C$

$f + 2i\sqrt{a}f - af$

$\ddot{f} = af$

$\ddot{g} = ag + C$

$g = f + \frac{C}{2}t^2$

$f + \frac{C}{2}t^2$

$\dot{f} + C$

$\ddot{f} = -af + C$

$\ddot{f} + af = C$

$\ddot{f} + 2\sqrt{a}\dot{f} + af = C + 2\sqrt{a}\dot{f}$

$e^{\sqrt{a}t}\ddot{f} + 2e^{\sqrt{a}t}\dot{f} + e^{\sqrt{a}t}f = Ce^{\sqrt{a}t} + 2\sqrt{a}\dot{f}e^{\sqrt{a}t}$

$\frac{d^2}{dt^2}(e^{\sqrt{a}t}f) = e^{\sqrt{a}}$

1D fluid flow (alternate model):

$\partial_t F = v$

$$\partial_t^2 F = -\partial_x^2 F + f$$

laws ∝

$\partial_t^2 \hat{F} = -4\pi^2 k^2 \hat{F} + \hat{f}$

$\partial_t v = -\partial_x p$ $\quad p_0 = \det(\mathbf{H} F_0)$

$\partial_t p = -\partial_x v$

$\partial_t^2 v = -\partial_x^2 v$

$\partial_t^3 F = -\partial_x^2 \partial_t F$

$\partial_t^2 F = -\partial_x^2 F + f(x)$

$x^2 + bx + c$

$x^2 + bx + \frac{b^2}{4} + c - \frac{b^2}{4}$

$(x + \frac{b}{2})^2 + c - \frac{b^2}{4}$

$\ddot{f} + af = C$

$\ddot{f} = -af + C$

$\dot{f} = \sqrt{-af^2 + Cf + D}$

$\int \frac{1}{\sqrt{-f^2 + f + D}} df = \sqrt{a}t$

$\dot{f} = \sqrt{2(-\frac{1}{2}af^2 + Cf) + D}$

$\int \frac{1}{\sqrt{f^2 + Cf + D}} df = \sqrt{-a}t$

$\int \frac{1}{\sqrt{(f+C)^2 + D - C^2}} df = \sqrt{-a}t$

$u = f + C$

$du = df$

$\int \frac{1}{\sqrt{u^2 + D}} du = \sqrt{-a}t$

$\int \frac{1}{\sqrt{D - u^2}} du = \sqrt{a}t$

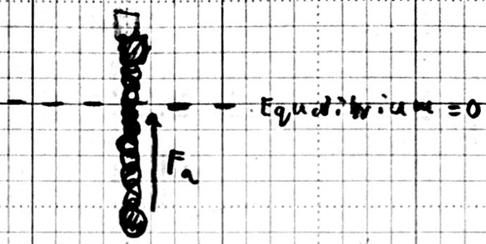
$\int \frac{\frac{1}{\sqrt{D}}}{\sqrt{1 - (\frac{u}{\sqrt{D}})^2}} du$ $\quad \int \frac{1}{\sqrt{1 - v^2}} dv$

$v = \frac{u}{\sqrt{D}}$

$dv = du \cdot \frac{1}{\sqrt{D}}$

$du = \sqrt{D}\, dv$

$\ddot{y} \propto -y$



$\ddot{y} \propto +y$

$\ddot{y} = -ky$

$\dot{y} = \sqrt{2\left(-\frac{k}{2}y^2 + C\right)}$

$\dot{y} = \sqrt{-ky^2 + C}$

$\dot{y} = \sqrt{k} \cdot \sqrt{C - y^2}$

$\int \frac{1}{\sqrt{C - y^2}} dy = \int \sqrt{k}\, dt$

$\int \frac{1}{\sqrt{C}} \cdot \frac{1}{\sqrt{1 - \left(\frac{y}{\sqrt{C}}\right)^2}} dy = \sqrt{k}\, t$

$u(x) = \frac{x}{\sqrt{C}}$

$\int \frac{du \cdot 1}{\sqrt{1-u^2}} \cdot \frac{du}{dy}\, dy = \sqrt{k}\, t$

$\int \frac{1}{\sqrt{1-u^2}} du = \sqrt{k}\, t$

$\sin^{-1}(u) + D = \sqrt{k}\, t$

$\sin^{-1}\left(\frac{y}{\sqrt{C}}\right) + D = \sqrt{k}\, t$

$\frac{y}{\sqrt{C}} = \sin(\sqrt{k}\, t)$

$y(t) = \sqrt{C} \sin(\sqrt{k}\, t + D)$
general solution

$y(t) = \sqrt{C} \sin(\sqrt{k}\, t + D)$

$y'(t) = \sqrt{kC} \cos(\sqrt{k}\, t + D)$

$y_0 = \sqrt{C} \sin(D)$

$V_0 = \sqrt{kC} \cos(D)$

$y_0^2 = C \sin^2(D)$

$\sin^2(D) = \frac{1 - \cos(2D)}{2}$

$2y_0^2 = C - C\cos(2D)$

$2y_0^2 - C = -C\cos(2D)$

$C - 2y_0^2 = C\cos(2D)$

$1 - \frac{2}{C} y_0^2 = \cos(2D)$

$D = \frac{\cos^{-1}\left(1 - \frac{2}{C} y_0^2\right)}{2}$

$2v_0^2 = kC + kC\cos(2D)$

$2v_0^2 = kC + kC\left(1 - \frac{2}{C} y_0^2\right)$

$2v_0^2 = kC + kC - 2k y_0^2$

$2v_0^2 = 2kC - 2k y_0^2$

$kC = v_0^2 + k y_0^2$

$C = \frac{v_0^2}{k} + y_0^2$

$1 - \frac{2}{C} y_0^2 = 1 - \frac{2}{\frac{v_0^2}{k} + y_0^2}$

~~$y(t) = \sqrt{\frac{v_0^2}{k} + y_0^2} \sin\left(\sqrt{k}\, t + \frac{\cos^{-1}\left(1 - \frac{2}{C} y_0^2\right)}{2}\right)$~~
~~potential (and very painful) taylored solution~~

$y(t) = \sqrt{\frac{v_0^2}{k} + y_0^2} \sin\left(\sqrt{k}\, t + \frac{1}{2}\cos^{-1}\left(1 - \frac{2}{\frac{v_0^2}{k} + y_0^2} y_0^2\right)\right)$
even more painful solution

$g:\{f:\mathbb{R}^2\to\mathbb{R}^1\}\to\mathbb{R}^1$

$g[f]$

$\nabla_f g[f]$

$$g[f]=\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy$$

$$\nabla_f g[f] := \frac{d}{dq}\int_{-1}^{1}\int_{-1}^{1} f(x,y)+q\,\delta(x-\alpha,y-\beta)\,dx\,dy\Big|_{q=0}$$

$$\boxed{\nabla_f g[f](\vec{\alpha}) := \partial_q\, g[f(\vec{x})+q\,\delta(\vec{x}-\vec{\alpha})]\Big|_{q=0}}$$

$$\frac{d}{dq}\int\int f(x,y)\,dx\,dy + \frac{d}{dq}\,q\int_{-1}^{1}\int_{-1}^{1}\delta(x-\alpha,y-\beta)$$

$$\frac{d}{dt}[f(t)+\delta(t-\alpha)] = f'(t)+\delta'(t-\alpha)$$

$$\boxed{\nabla_f g[f]=1}$$

and this works too!

$f:\mathbb{R}^1\to\mathbb{R}^1$

$$\nabla_f g[f,f'] := \partial_q g[f(t)+\delta(t-\alpha),\, f'(t)+\delta'(t-\alpha)]\Big|_{q=0}$$

$$A(x)=\int_a^b L(x,\dot{x})\,dt$$

$$\nabla_x A(x)\equiv 0$$

~~$\partial_q\int_a^b L(x(\tau)+q\,\delta(\tau-t),\dot{x}(\tau)+q\,\dot{\delta}(\tau-t))\,d\tau$~~

stationary point: $\nabla_f g[f]\equiv 0$

$t\in[a,b]$

$$\partial_q\int_a^b L(x(\tau)+q\,\delta(\tau-t),\,\dot{x}(t)+q\,\dot{\delta}(\tau-t))\,d\tau\Big|_{q=0}$$

$$\int_a^b f(\tau)\,\delta'(\tau-x)\,d\tau = -f'(x)$$

$$\int_a^b \frac{\partial L}{\partial x}\cdot\delta(\tau-t)+\frac{\partial L}{\partial \dot{x}}\cdot\dot{\delta}(\tau-t)\,d\tau=0$$

$$\lim_{h\to 0}\int_a^b \frac{f(\tau)\,\delta(\tau-x+h)-f(\tau)\,\delta(\tau-x)}{h}$$

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0$$

$$\lim_{h\to 0}\frac{f(x-h)-f(x)}{h} = -f'(x)$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=\frac{\partial L}{\partial x}}$$

yippy! It works!