Modeling Relativistic Strings as Level Curves

Max Orton

Fall 2024

My work with Relativistic Strings is my most recent project. In The Theoretical Minimum recorded lectures on String Theory, Susskind used parameterized strings, and worked in the lightcone frame. I wanted to create a manifestly relativistic model that could handle multiple strings at the same time. This write-up is the result.

Our goal in the following is to find a way to describe 1+1 dimensional structures consistent with special relativity. We will begin in 2+1 dimensions, then generalize to higher dimensions and discuss possible ways to quantize our system.

Our starting point will be the fact that for a collection of relativistic strings, the action is proportional to the total surface area of the collection. Though later we will find a more explicit form for the action, for now we can write:

$$S = kA$$

Where S is the action, A is the total surface area, and k is a constant to maintain dimensional consistency. It's worth noting that k will need units of action divided by area which means that k is a force or, in the same vein, a tension. This works because, though area normally has units of length squared, a string moving slower than light will instead have its area measured in units of length time. When I first noticed this, I was very excited, but honestly, I think this result might just be a coincidence. We will encounter a better physical interpretation for k later on.

There is only so much we can do with the action written in so abstract a form, so our next step will be to write it explicitly in terms of the model we are using to describe the strings. We will be modeling the strings as the level curves of, in general, D-1 functions in D+1 dimensional space. The advantage to this approach is that multiple strings and even interactions between strings are baked into the model's structure and do not have to be added later.

2+1 dimensional space is the simplest, non-trivial example, so we will begin there. To do this, we will define a function $\phi(X)$ such that at each point X in space time where a string exists:

$$\phi(X) = 0$$

Our next goal will then be to find the surface area of the level curves of this function. We only want points where the function is zero to contribute, so we can try including a delta function of the field. We multiply this by an as yet undetermined function of the derivatives of the field. It is only a function of the derivatives of ϕ, because everywhere the integrand is non-zero, the field is zero.

$$A[\phi] = \int_{t_i}^{t_f} \int_{\Omega} \Lambda(\partial\phi)\delta(\phi)\, dA\, dt$$

We can now try to find the form of Λ. If the field didn't change over time ($\dot{\phi} = 0$), then our integral would simplify:

$$A[\phi] = (t_f - t_i) \int_{\Omega} \Lambda(\nabla\phi)\delta(\phi) dA$$

The integral is evaluated now only on a single time slice and can be seen as the total length of the collection of strings on that slice. That, multiplied by our change in time ($t_f - t_i = \Delta t$), gives us the total relativistic surface area of the strings.

Next, we will break up space into many small regions $\{\Omega_n\}$ and assign each one a point $\vec{x_n} \in \Omega_n$. Each of the Ω_n is picked to be small enough that ϕ is approximately linear within the region.

$$\phi(\vec{x}) \approx \phi(\vec{x_n}) + (\vec{x} - \vec{x_n}) \cdot \vec{\nabla}\phi(\vec{x_n}) \quad \forall x \in \Omega_n$$

We can rewrite our integral as a sum over the individual regions. We will also define shifted integration variables $\vec{y} = \vec{x} - \vec{x_n}$.

$$A[\phi] = \Delta t \sum_n \int_{\Omega_n} \Lambda(\nabla\phi(x_n + y))\delta(\phi(\vec{x_n}) + \vec{y} \cdot \vec{\nabla}\phi(\vec{x_n})) dy_1 dy_2$$

We can now evaluate each of these integrals separately. We will invent a new pair of coordinates that are rotated such that one, $y_\parallel$, is parallel to $\vec{\nabla}\phi$ and the other,$y_\perp$, is perpendicular to it. Since all we did was rotate the coordinates in space, $dy_1 dy_2 = dy_\parallel dy_\perp$. The dot product of y with $\vec{\nabla}\phi$ is simply $y_\parallel \|\vec{\nabla}\phi\|$. Plugging this in, we get:

$$A[\phi] = \Delta t \sum_n \int_{\Omega_n} \Lambda(\nabla\phi(x_n + y))\delta(\phi(\vec{x_n}) + y_\parallel \|\vec{\nabla}\phi\|) dy_\parallel dy_\perp$$

Carrying out the integral over $y_\parallel$ yields:

$$A[\phi] = \Delta t \sum_n \int \frac{\Lambda(\nabla\phi(x_n + y))}{\|\vec{\nabla}\phi\|} dy_\perp \,|_{y_\parallel = 0}$$

The integral is performed along the small segment of the curve $\phi = 0$ contained inside Ω_n. Adding all our regions back together this gives us a line integral over the curve $\phi = 0$.

$$A[\phi] = \Delta t \oint_{\phi(\vec{x})=0} \frac{\Lambda(\nabla\phi)}{\|\vec{\nabla}\phi\|} ds$$

If we want the integral to represent the length of the curve, the integrand should clearly be 1. This finally allows us to pin down a form for Λ!

$$\Lambda(\nabla\phi) = \|\vec{\nabla}\phi\|$$

This gives us an explicit action for the case where $\dot{\phi} = 0$:

$$S[\phi] = \int k\Delta t \|\vec{\nabla}\phi\| \delta(\phi) d^2x$$

Finding the general form turns out to be surprisingly easy. There is really only one way to write a relativistically invariant function of $\partial_\mu\phi$ that reduces to $\|\vec{\nabla}\phi\|$ when $\dot{\phi} = 0$. Here, we are using the mostly negative form of $\eta_{\mu\nu}$.

$$\sqrt{\|\vec{\nabla}\phi\|^2 - \dot{\phi}} \quad \text{or equivalently} \quad \sqrt{-\partial_\mu\phi\partial^\mu\phi}$$

Plugging this in, we've found the general form for our action!

$$S[\phi] = \int k\sqrt{-\partial_\mu\phi\partial^\mu\phi}\,\delta(\phi) dX$$

Before we dive into exploring the properties of this action, we'll do two quick example calculations to see if they give reasonable results. The first example is that of a string stretched along the x_2-axes from $-\infty$ to $+\infty$ and moving along the x_1 axes. Since our x_1 coordinate is changing as a function of time, we can write:

$$X(t) = x_1 \quad \text{or} \quad 0 = X(t) - x_1$$

This has the form $\phi(x_1, t) = 0$ which means that $X(t) - x_1$ can serve as ϕ in our model. All we need to do to find $X(t)$ is plug it into our formula for the action.

$$-\partial_\mu \phi \partial^\mu \phi = 1 - \dot{X}^2$$

$$S[X] = \int k\sqrt{1 - \dot{X}^2}\, \delta(X(t) - x_1) dx_1 dx_2 dt$$

Integrating over x_1 and moving around our integral over x_2 we get:

$$S[X] = \int \left(\int k\, dx_2 \right) \sqrt{1 - \dot{X}^2}\, dt$$

This is just the action for a relativistic particle of total mass $-\int k\, dx_2$! This gives us our better physical meaning for the constant k. It is the negative of the mass density of the strings $k = -\mu$. From here on, I will be writing the action in terms of μ instead of k as it has more physical meaning.

We will do one more slightly less trivial example: that of a circular string. We will do something very similar to what we did before when defining $X(t)$, but now we will define $R(t) = \sqrt{x_1^2 + x_2^2} = r$ which implies $\phi(r, t) = R(t) - r$. We can then plug this in to find the action. I have skipped the intermediate steps and jumped right to the final result.

$$S[R] = \int -\mu R\sqrt{1 - \dot{R}^2}\, dt$$

We can then use the Euler Lagrange Equations to find R.

$$\frac{d}{dt}\left(\frac{R\dot{R}}{\sqrt{1 - \dot{R}^2}} \right) + \sqrt{1 - \dot{R}^2} = 0$$

When I first tried this problem, I used conservation of energy to make it first order, then went about solving it mechanically. It was anything but easy. The simpler way is to notice that the form of $\sqrt{1-\dot{R}^2}$ seems to lend itself to a solution in terms of sine or cosine. I have since come up with a third way of solving this, by far my favorite, which includes a Wick rotation and the fact that hanging strings make hyperbolic cosine graphs. This is included in my video. For now we'll go with the second method, plug in $R = \alpha \sin(\beta t + \gamma)$ and see what we get (I'm defining $\theta = \beta t + \gamma$ for the sake of fitting the equation on the page).

$$\frac{d}{dt}\left(\frac{\alpha^2 \beta \sin(\theta)\cos(\theta)}{\sqrt{1-\alpha^2\beta^2\cos(\theta)^2}}\right) + \sqrt{1-\alpha^2\beta^2\cos(\theta)^2} = 0$$

This is satisfied if $\alpha\beta = 1$. Thus we have:

$$R(t) = \alpha \sin\left(\frac{t}{\alpha} + \gamma\right)$$

This is a two parameter family of solutions to a second order differential equation, so we can be reasonably confident it represents the entire solution set. There are a few interesting things to note about this solution. First of all, its physical characteristics are determined by only a single number α: the maximum radius. The other constant, γ, is a purely mathematical construct related to when we start counting time.

Second of all, it is pretty clear that the solution only makes sense for $t \in [-\alpha\gamma, \alpha(2\pi + \gamma)]$. Outside of that interval, our solution gives a negative radius. At the point when the radius equals zero, the string's inward velocity $\dot{R}$ approaches the speed of light. This is a kind of singularity. The total energy of the string must stay constant, while the radius (and thus the length) of the string approaches zero. This means that the energy density of the string blows up to infinity as the string shrinks.

Over the last month or so I've been trying to show that this singularity happens for all collections of strings of finite size in 2+1 dimensional space, or that there are some configurations that avoid it. For now, I haven't been able to show either. My attempt has centered around checking whether the area encompassed by the collection of strings must shrink to zero. The change in area does have a very simple form: the integral of $\dot{\phi}\delta(\phi)$ over all space. I haven't yet been able to show what this implies.

For now, we can use the Euler Lagrange equations to find a general equation of motion for our system.

$$\frac{\partial}{\partial X^\mu}\frac{\partial \mathcal{L}}{\partial\left[\frac{\partial\phi}{\partial X^\mu}\right]}=\frac{\partial \mathcal{L}}{\partial\phi}$$

$$\frac{\partial}{\partial X^\mu}\left[\frac{-\partial_\mu\phi\,\delta(\phi)}{\sqrt{-\partial_\mu\phi\partial^\mu\phi}}\right]=\sqrt{-\partial_\mu\phi\partial^\mu\phi}\,\delta'(\phi)$$

Amazingly, when we expand it, all the terms multiplying $\delta'(\phi)$ cancel, leaving:

$$[\partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu\phi-\partial_\sigma\phi\partial_\sigma\phi\partial^\tau\partial_\tau\phi]\delta(\phi)=0$$

There are multiple ways to satisfy this equation. Perhaps the easiest is to constrain ϕ more, restricting ourselves to ϕ for which the part multiplying the delta function is zero at all points, not just when the delta function itself is non-zero. This is still a second order, non-linear partial differential equation. It is anything but easy to solve.

So far, we have only studied strings in 2+1 dimensions but I'll conclude by generalizing our action to D+1 dimentional space. The action will now be a functional of D-1 functions $\{\phi_n\}$. The string exists at all points X where:

$$\phi_n(X)=0 \quad \forall n$$

Just like before, we only want points where the string exists to contribute to the action, so we will write it as a delta function multiplied by a function of the derivatives of all the ϕ_n.

$$S=\int_{t_i}^{t_f}\int_\Omega \Lambda(\{\partial\phi_n\})\prod_n \delta(\phi_n)\,dx_i\,dt$$

Like last time, we first consider the case where the string is stationary ($\dot{\phi}\equiv 0 \;\; \forall n$).

$$S=\Delta t\int_\Omega \Lambda(\{\nabla\phi_n\})\prod_n \delta(\phi_n)\,dx_i$$

Then, we break up omega into many sub-regions Ω_α (small enough that all the ϕ_α are approximately linear over the regions) and pick corresponding points $x_\alpha\in\Omega_\alpha$, then split up our integral.

$$S=\Delta t\sum_\alpha\int_{\Omega_\alpha}\Lambda(\{\nabla\phi_n(x)\})\prod_n \delta(\phi_n(x))\,dx_i$$

Like last time, we define new integration variables in each of our regions $(\{y_{\|n}\}, y_{\perp})$. These are shifted so that the new origin corresponds to x_{α}, and all but one of the unit vectors (unit both in the new and old system) point along the D-1 gradients of the ϕ_n, while the last one is perpendicular to all of them. We can write our new volume element in terms of the derivatives of ϕ where ϵ is the completely antisymmetric tensor.

$$dx = \frac{\partial\{x_i\}}{\partial\{y_i\}} dy_{\|}\, dy_{\perp} = \frac{\prod_n \|\nabla\phi_n\|}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}} dy_{\|}\, dy_{\perp}$$

Next we will plug this into our integral and simultaneously make the assumption that all the ϕ_n are approximately linear in the regions Ω_{α}. This gives us:

$$S = \Delta t \sum_{\alpha} \int_{\Omega_{\alpha}} \Lambda(\{\nabla\phi_n(x)\}) \prod_n \delta(\|\nabla\phi_n\| y_{\|n}) \frac{\prod_n \|\nabla\phi_n\|}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}}\, dy_{\|}\, dy_{\perp}$$

Integrating over all of the $y_{\|n}$ cancels the product in the numerator of our volume element, giving us:

$$S = \Delta t \sum_{\alpha} \int_{\Omega_{\alpha}, \phi=0} \frac{\Lambda(\{\nabla\phi_n(x)\})}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}}\, dy_{\perp}$$

Each of these is an integral over the small and approximately straight segment of string contained in each of the Ω_{α}. I haven't explicitly written it out, but if there was no point in a given Ω_{α} where $\phi_n = 0\ \forall n$, then the integrand for that Ω_n would instead be zero. Like last time, adding all of the Ω_{α} back together yields a line integral over the strings.

$$S = \Delta t \oint_{\phi(\vec{x})=0} \frac{\Lambda(\{\nabla\phi_n\})}{\sum_{\{i_n\}} \epsilon^{i_1 i_2 i_3 \cdots} \prod_n \frac{\partial\phi_n}{\partial x^{i_n}}}\, ds$$

If we want this to be proportional to the total length of all the strings, the integrand needs to be constant. We can write it in terms of μ, the mass density of the string. This finally lets us pin down a form for Λ!

$$\Lambda(\{\nabla\phi_n\}) = -\mu \sum_{\{i_n\}} \epsilon_{\{i_n\}} \prod_n \frac{\partial\phi_n}{\partial x_{i_n}}$$

We can plug this into our action in terms of Λ to find the action of a stationary string.

$$S = \Delta t \int_{\Omega} -\mu \sum_{\{i_n\}} \epsilon_{\{i_n\}} \prod_{n} \frac{\partial \phi_n}{\partial x_{i_n}} \delta(\phi_n) d^3x$$

It's much less obvious than last time, but there is still basically only one relativistically invariant function of the derivatives of the ϕ_n that reduces to the above integrand when $\dot{\phi}_n = 0$ (The first picture on my website is me showing how this reduces to the above action for a stationary string). Substituting this into our integral, we've found the total action for a collection of relativistic strings in D+1 dimensional space!

$$S = \int_{t_i}^{t_f} \int_{\Omega} -\mu \sqrt{\epsilon^{\mu\{\beta_n\}} \epsilon_{\mu\{\alpha_n\}} \prod_{n}^{D-1} \frac{\partial \phi_n}{\partial x_{\alpha_n}} \frac{\partial \phi_n}{\partial x^{\beta_n}}} \prod_{n}^{D-1} \delta(\phi_n) d^3x$$

If we solve for our equations of motion using the Euler Lagrange equations, we again find that all the terms containing $\delta'(\phi_n)$ cancel, and we get a single equation multiplied by a delta function of the field.

$$\forall j \quad \left[\prod_{n} \delta(\phi_n)\right] \frac{\partial}{\partial x_{\alpha_j}} \left[\frac{\epsilon^{\mu\{\beta_n\}} \epsilon_{\mu\{\alpha_n\}} \left(\prod_{n \neq j}^{D-1} \frac{\partial \phi_n}{\partial x_{\alpha_n}} \frac{\partial \phi_n}{\partial x^{\beta_n}} \right) \frac{\partial \phi_j}{\partial x^{\beta_j}}}{\sqrt{\epsilon^{\mu\{\beta_n\}} \epsilon_{\mu\{\alpha_n\}} \prod_{n}^{D-1} \frac{\partial \phi_n}{\partial x_{\alpha_n}} \frac{\partial \phi_n}{\partial x^{\beta_n}}}} \right] = 0$$

If we mandate that everywhere, not just $\phi_n = 0$, the field follows this equation, we can write the evolution of ϕ as a field equation with no delta functions. It's worth remarking how amazing it is that for all dimensions, terms containing derivatives of δ cancel. This was far from what I expected. So far this is the only action I've found that includes a delta function where this happens. Proving that this is the only action where the higher order derivatives of delta functions cancel is another project I've been working on.

Though everything written here was done classically, in the future I hope to quantize this model. I've been working on this from two angles. The first is to plug the action (delta function and all) into the path integral. The second is to treat the field equation I derived as a quantum field, then look for level curves using a delta function of the field operator. I've also been looking into representing higher dimensional shapes and open strings with similar models.