Perturbative Expansion for Interacting Quantum Field Theories

Max Orton

Summer 2024

The work here is, in many ways, an extension of my quantization of free fields. Though my original method worked well with linear fields, it couldn't handle interactions, so I spent a year developing this perturbative approach. I find it amazing that this, combined with my quantization of linear fields, describes nearly all the matter within and around us.

Given the free field evolution for a set of bosonic and fermionic fields and a form for the interaction terms in the action, our goal to predict the evolution of the fields.

The amplitude that an initial state $|\psi_I\rangle$ will transform into a final state $|\psi_F\rangle$ after an elapsed time T is:

$$\langle\psi_F, T|\psi_I\rangle = \langle\psi_F|\, e^{-iTE}\, |\psi_I\rangle$$

In order to explore theories where only one part of the Hamiltonian is known we make the following definition:

$$E = E_F(0) - L_I(0)$$

Here, E_F governs the free evolution of the system, while L_I is the interacting terms in the Lagrangian. The total Hamiltonian E is time independent, but in general, E_F and L_I will not be conserved, and so we have arbitrarily chosen to evaluate them at time 0.

We can next use one definition of e^x to expand our time evolution:

$$\langle\psi_F| e^{-iTE} |\psi_I\rangle = \lim_{\epsilon\to 0} \langle\psi_F| [1 - i\epsilon E_F(0) + i\epsilon L_I(0)]^{(\frac{T}{\epsilon})} |\psi_I\rangle$$

We can now expand this to increasing order in the interacting Lagrangian. Our first term, $\mathcal{O}(L_I^0)$, will of course simply be:

$$\lim_{\epsilon\to 0} \langle\psi_F| [1 - i\epsilon E_F(0)]^{(\frac{T}{\epsilon})} |\psi_I\rangle = \langle\psi_F| e^{-iTE_F(0)} |\psi_I\rangle$$

This just describes evolution in the free theory, the details of which we already know. Next, we will consider the terms in the expansion that contain one power of L_I. These will be characterized by a single number detailing which factor in the product it was pulled from. We can call this number n and associate it with a time $t \in [0, T]$ where $t = \epsilon n$. Using t gives us one distinct advantage: while n depends on our choice of discretization, t does not, so it will still make sense after ϵ tends to zero. This second term in our expansion will have the following form:

$$\lim_{\epsilon\to 0} \sum_n \langle\psi_F| (1 - i\epsilon E_F(0))^{\frac{T}{\epsilon}-n} iL_I(0)(1 - i\epsilon E_F(0))^{n-1} |\psi_I\rangle$$

Taking the limit as ϵ tends towards 0, and remembering our definition of t, we get:

$$i\int_0^T \langle\psi_F| e^{-iTE_F(0)} \left[e^{itE_F(0)} L_I(0) e^{-itE_F(0)}\right] |\psi_I\rangle\, dt$$

The important thing to note is that the section to the left of the brackets is ψ_F shifted to time T in the free theory. Likewise, the section inside the brackets is the Heisenberg formula for operator evolution evaluated on L_I and thus is $L_I(t)$. Bringing this all together we get:

$$\langle\psi_F, T| i\int_0^T L_I(t)dt\, |\psi_I, 0\rangle \;\text{ or }\; \langle\psi_F, T| iS_I |\psi_I, 0\rangle$$

Here, S_I is the interacting part of the action. It's worth noting that S_I is the interacting part of the action for the interval of time we are studying, so is integrated only from the initial to final time.

The derivation of the remaining terms runs along the exact same lines, so I'll only include the results below. Note that $\mathcal{T}$ denotes the time ordered product and everything is evaluated in the free theory:

$$\mathcal{O}(L_I^2) : \int_0^T \int_0^{t_2} \langle \psi_F, T| \left[iL_I(t_2) \right] \left[iL_I(t1) \right] |\psi_I, 0\rangle \, dt_1 dt_2 = \langle \psi_F, T| \, \mathcal{T} \left\{ \frac{(iS_I)^2}{2} \right\} |\psi_I, 0\rangle$$

$$\mathcal{O}(L_I^3) : \langle \psi_F, T| \, \mathcal{T} \left\{ \frac{(iS_I)^3}{3!} \right\} |\psi_I, 0\rangle$$

$$\mathcal{O}(L_I^4) : \langle \psi_F, T| \, \mathcal{T} \left\{ \frac{(iS_I)^4}{4!} \right\} |\psi_I, 0\rangle$$

The general form is clearly:

$$\mathcal{O}(L_I^n) : \langle \psi_F, T| \, \mathcal{T} \left\{ \frac{(iS_I)^n}{n!} \right\} |\psi_I, 0\rangle$$

Adding all of them up and using the power series for e^x we find a form for the amplitude. Since everything is evaluated in the *free* theory, we know how to calculate it!

$$\langle \psi_F, T| \, \mathcal{T} \left\{ e^{iS_I} \right\} |\psi_I, 0\rangle$$

I've been searching for a way to quantize interacting fields since hearing about the Feynman calculus in one of Susskind's recorded lectures two years ago. Working this out was one of the most exciting moments of the last few years for me. I also figured out the Feynman propagator as a time ordered product of fields at around the same time, but didn't have time to write it into this portfolio. I haven't yet been able to do many calculations with this method. This is because I'm fairly certain the vacuum state is different in interacting and free theories, which has kept me from correctly specifying initial and final conditions. I'm working on a way to fix that. In the meantime, it's incredible to be able to plug a temporary source field (eg; a lightbulb) into the vacuum and calculate on average how many particles it produces and in what states!