Deriving the Canonical Quantization Approach from the Path Integral

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Spring 2024

My work with the path integral was another attempt to make quantum mechanics consistent with special relativity. It took me several attempts to find its correct form as I began this work knowing only that the amplitude involved a sum over paths weighted by a complex exponential of the action.

Our goal is to start with the basic principles of the path integral approach and use them to derive canonical quantization. We will be studying an n particle system with positions and momenta $\{q_i, p_i\}$. Each state is characterized by a wave function $\psi(\{q_i\})$. For our purposes, it is a function only of the $\{q_i\}$ as we will work only in position space. The amplitude that state ψ_I will transition to state ψ_F over elapsed time $t_I \to t_F$ is given by:

$$\langle \psi_F, t_F | \psi_I, t_I \rangle \propto \int \overline{\psi}_F(\{q_i(t_F)\})\psi_I(\{q_i(t_I)\})e^{iS[\{q_i\}]}\mathcal{D}q_i$$

Note that for our purposes, $S[\{q_i\}]$ is the action that exists between t_I and t_F.

$$S[\{q_i\}] = \int_{t_I}^{t_F} \mathcal{L}(\{q_i(t)\})dt$$

First, we will find the form of the momentum operator. We can do this by multiplying it by two vectors, ψ_1 and ψ_2 on the right and left respectively. We have to shift ψ_2 forward by an infinitesimal time shift ϵ because momentum doesn't make sense in the path integral without some amount of elapsed time.

Later, we'll let it tend to zero.

$$\langle \psi_2, t | \, p_i \, | \psi_1, t \rangle \propto \lim_{\epsilon \to 0} \int \overline{\psi}_2(\{q_i(t+\epsilon)\})\psi_1(\{q_i(t)\})p_i(t) \exp\left\{ i \int_t^{t+\epsilon} \mathcal{L}(\tau) d\tau \right\} \mathcal{D}q_i$$

We can solve this integral by discritizing time. We will break it down into steps of size epsilon which effectively makes our path a straight line from the initial point to the final point. Amazingly, this is enough resolution to find the momentum operator. With this restriction on our path $q(t) = q$, $q(t+\epsilon) = q + \epsilon \dot{q}$ and $S = \epsilon \mathcal{L}$. We will also use the fact that the cannonical momentum to q_i is $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$. Plugging all this in, we get:

$$\langle \psi_2, t | \, p_i \, | \psi_1, t \rangle \propto \lim_{\epsilon \to 0} \int \overline{\psi}_2(\{q_i + \epsilon \dot{q}_i\})\psi_1(\{q_i(t)\})p_i(t) \exp\{ i\epsilon \mathcal{L}(t) \} \, dq_i d\dot{q}_i$$

We can rewrite p_i as coming from a derivative of the exponential,

$$\langle \psi_2, t | \, p_i \, | \psi_1, t \rangle \propto \lim_{\epsilon \to 0} \frac{-i}{\epsilon} \int \overline{\psi}_2(\{q_i + \epsilon \dot{q}_i\})\psi_1(\{q_i(t)\}) \frac{\partial}{\partial \dot{q}_i} \exp\{ i\epsilon \mathcal{L}(t) \} \, dq_i d\dot{q}_i$$

integrate by parts and do the chain rule,

$$\langle \psi_2, t | \, p_i \, | \psi_1, t \rangle \propto \lim_{\epsilon \to 0} i \int \frac{\partial}{\partial q_i} \left(\overline{\psi}_2(\{q_i + \epsilon \dot{q}_i\}) \right) \psi_1(\{q_i(t)\}) \exp\{ i\epsilon \mathcal{L}(t) \} \, dq_i d\dot{q}_i$$

let epsilon go to zero,

$$\langle \psi_2 | \, p_i \, | \psi_1 \rangle \propto i \int \frac{\partial}{\partial q_i} \left(\overline{\psi}_2(\{q_i\}) \right) \psi_1(\{q_i(t)\}) dq_i$$

and integrate by parts again.

$$\langle \psi_2 | \, p_i \, | \psi_1 \rangle \propto \int \overline{\psi}_2(\{q_i\}) \left[-i \frac{\partial}{\partial q_i} \right] \psi_1(\{q_i(t)\}) dq_i$$

Up to a constant of proportionality, we've found the momentum operator!

Our next goal is to show that states, evolve by repeated application of $(1 - i\epsilon\mathcal{H})$ where $\mathcal{H}$ is the Hamiltonian. To do this, we will take the inner product of two states separated by a small time step δt. In this case, $\delta t = 2\epsilon$, where epsilon is the size of the discretization we will use to solve the path integral. Finding the energy operator requires this slightly longer path. We will be trying to show:

$$\langle\psi_2, t+2\epsilon|\psi_1, t\rangle = \langle\psi_2, t+\epsilon|\,(1-i\epsilon\mathcal{H})\,|\psi_1, t\rangle$$

Our first step is to write out the inner product as a path integral.

$$\langle\psi_2, t+2\epsilon|\psi_1, t\rangle \propto \int \overline{\psi}_2(\{q_i(t+2\epsilon)\})\psi_1(\{q_i(t)\})\exp\left(i\int_t^{t+2\epsilon}\mathcal{L}(\tau)d\tau\right)\mathcal{D}q$$

Again, we will discretize time into steps of size $\epsilon = \frac{1}{2}\delta t$. We than make the following substitutions:

1: $q_i(t+n\epsilon) = q_n^i$
1: $\dot{q}_i(t+n\epsilon) = \dot{q}_{n,n+1}^i$
2: $\mathcal{L}(t+n\epsilon) = \mathcal{L}_{n,n+1}$
3: $S = \epsilon\mathcal{L}_{01} + \epsilon\mathcal{L}_{12}$

Now we can rewrite our integral in terms of these new variables:

$$\langle\psi_2, t+2\epsilon|\psi_1, t\rangle \propto \int \overline{\psi}_2(\{q_1^i + \epsilon\dot{q}_{12}^i\})\psi_1(\{q_0^i\})\exp\left(i\epsilon\mathcal{L}_{01} + i\epsilon\mathcal{L}_{12}\right)dq_0^i\,d\dot{q}_{01}^i\,dq_{12}^i$$

Now we expand to first order in epsilon. We have to do this carefully because we only want to expand around the second time step. Thus, we will leave the $\exp(i\epsilon\mathcal{L}_{01})$ untouched. The first order term is:

$$\epsilon\left(\int\left[i\overline{\psi}_2(\{q_1^i\})\mathcal{L}_{12} + \sum_i\left(\frac{\partial\overline{\psi_2}}{\partial q_i}\dot{q}_{12i}\right)\right]\psi_1(\{q_0^i\})\exp\left(i\epsilon\mathcal{L}_{01}\right)dq_0^i\,d\dot{q}_{01}^i\,dq_{12}^i\right)$$

We can break this into two separate integrals and evaluate them individually before adding them back together. By far the simpler of the two is the first term in the brackets above (the one multiplied by the Lagrangian between times one and two).

$$i\epsilon\int\mathcal{L}_{12}\overline{\psi}_2(\{q_1^i\})\psi_1(\{q_0^i\}))\exp(i\epsilon\mathcal{L})dq_0^i\,d\dot{q}_{01}^i\,dq_{12}^i$$

This term doesn't require much manipulation. All we need to do is rewrite it to have the same coefficient as the Hamiltonian in the time evolution operator. This gives us the negative of the Lagrangian which is, indeed, a term in the Hamiltonian!

$$-i\epsilon \int (-\mathcal{L}_{12})\overline{\psi}_2(\{q_1^i\})\psi_1(\{q_0^i\}))\exp(i\epsilon\mathcal{L}_{01})dq_0^i\, d\dot{q}_{01}^i\, dq_{12}^i$$

The other term will require a little more manipulation. This will amount to the inverse of what we did with the momentum operator. There, we took a factor of momentum in the integrand and replaced it with a derivative. Here, we will take a derivative and replace it with a factor of the momentum in the integrand. We do the following steps:

$$\epsilon \sum_i \int \dot{q}_{12i} \frac{\partial}{\partial q_{0i}} \overline{\psi}_2(\{q_{0i} + \epsilon \dot{q}_{01i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

We can replace the derivative with respect to q_{0i} with a derivative with respect to $\dot{q}_{01i}$, which cancels the ϵ out front.

$$\sum_i \int \dot{q}_{12i} \frac{\partial}{\partial \dot{q}_{01i}} \overline{\psi}_2(\{q_{0i} + \epsilon \dot{q}_{01i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

Then, we integrate by parts, shifting the derivative from $\overline{\psi}_2$ onto the rest of the integrand (at the cost of a negative sign). The only thing besides $\overline{\psi}_2$ that depends on $\dot{q}_{01i}$ is $\mathcal{L}_{01}$. This gives us:

$$-\sum_i \int \dot{q}_{12i} \overline{\psi}_2(\{q_{0i} + \epsilon \dot{q}_{01i}\})\psi_1(\{q_{0i}\})\frac{\partial}{\partial \dot{q}_{01i}}\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

Evaluating this derivative, we pull down a factor of i, a factor of ϵ, and a derivative of $\mathcal{L}_{01}$ with respect to $\dot{q}_{01i}$, which we recognize as the conjugate momentum p_{01i}!

$$\int \left(-i\epsilon \sum_i \dot{q}_{12i} p_{01i}\right) \overline{\psi}_2(\{q_{1i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

Adding these back together, we have an expression for our original inner product (where k is a constant of proportionality that normalizes our path integral).

$$\langle\psi_2, t+\epsilon|\psi_1, t\rangle - ik\epsilon \int \left(\sum_i \dot{q}_{12i} p_{01i} - \mathcal{L}_{12}\right) \overline{\psi}_2(\{q_{1i}\})\psi_1(\{q_{0i}\})\exp(i\epsilon\mathcal{L}_{01})\mathcal{D}q$$

But... the expression in parenthesis is just the Hamiltonian! Rewriting this all in terms of inner products, we've derived the expression for the evolution of a quantum state in the *canonical quantization* approach only using the *path integral*!

$$\langle\psi_2, t + 2\epsilon|\psi_1, t\rangle = \langle\psi_2, t + \epsilon|\,(1 - i\epsilon\mathcal{H})\,|\psi_1, t\rangle$$

There is a lot I plan to do with the path integral approach in the future. For instance, I have hopes to recast the work I've done in quantum field theory in terms of the path integral. The way I've written it so far explicitly singles out time. I hope to eventually remove this, perhaps by creating a way to measure and restrict the amount of information stored in a given state.